Effective Resistance and Capacitance in Simplicial Complexes and a Quantum Algorithm

Mitchell Black ✉
Oregon State University, Corvallis, OR, USA

William Maxwell ✉
Oregon State University, Corvallis, OR, USA

─── **Abstract** ───

We investigate generalizations of the graph theoretic notions of effective resistance and capacitance to simplicial complexes and prove analogs of formulas known in the case of graphs. In graphs the effective resistance between two vertices is $O(n)$; however, we show that in a simplicial complex the effective resistance of a null-homologous cycle may be exponential. This is caused by relative torsion in the simplicial complex. We provide upper bounds on both effective resistance and capacitance that are polynomial in the number of simplices as well as the maximum cardinality of the torsion subgroup of a relative homology group denoted $\mathcal{T}_{\max}(\mathcal{K})$. We generalize the quantum algorithm deciding st-connectivity in a graph and obtain an algorithm deciding whether or not a $(d-1)$-dimensional cycle γ is null-homologous in a d-dimensional simplicial complex $\mathcal{K}$. The quantum algorithm has query complexity parameterized by the effective resistance and capacitance of γ. Using our upper bounds we find that the query complexity is $O\left(n^{5/2} \cdot d^{1/2} \cdot \mathcal{T}_{\max}(\mathcal{K})^2\right)$. Under the assumptions that γ is the boundary of a d-simplex (which may or may not be included in the complex) and that $\mathcal{K}$ is relative torsion-free, we match the $O(n^{3/2})$ query complexity obtained for st-connectivity. These assumptions always hold in the case of st-connectivity. We provide an implementation of the algorithm whose running time is polynomial in the size of the complex and the relative torsion. Finally, we prove a duality theorem relating effective resistance and capacitance when $\mathcal{K}$ is d-dimensional and admits an embedding into $\mathbb{R}^{d+1}$.

1 Introduction

The *effective resistance* between two vertices s and t in a graph is a quantity that measures how "well-connected" s and t are; specifically, the more, shorter paths connecting s and t, the lower the effective resistance between s and t. While effective resistance was originally defined in the context of resistor networks, it has since been discovered that effective resistance has many other interpretations. It is a metric on the vertices of a graph [17]; it is proportional to the expected number of steps in a random walk from s to t and back to s [3]; if $\{s, t\}$ is an edge, it is propotional to the probabilty $\{s, t\}$ is in a random spanning tree of G [15]. Effective resistance also has applications. Sampling edges in a graph with probability proportional to the effective resistance between their endpoints produces a graph that approximates the spectrum of the Laplacian of the original graph [27]. Effective resistance is also a parameter in the running time of a quantum algorithm to decide if s and t are connected [1].

Recently, effective resistance has independently been generalized from graphs to simplicial complexes by several groups of authors, although each group defines effective resistance for a different object or objects in a simplicial complex. Kook and Lee [18] and Osting,

Palande, and Wang [23] both define effective resistance as a quantity associated with the boundary of a simplex.[1] This definition is analogous to the graph definition, as a pair of vertices is the boundary of an edge. Hansen and Ghrist [9] give a more general definition and define effective resistance as a quantity between two homologous cycles[2]. Our definition is equivalent to Hansen and Ghrist's, as we define effective resistance as a quantity associated with a null-homologous cycle.

Recent research has shown that some properties of effective resistance in graphs generalize to simplicial complexes. For example, Kook and Lee prove that the effective resistance of the boundary a simplex is the probability that the simplex is in the high-dimensional equivalent of a spanning tree [18]. Osting, Palande, and Wang show that sampling d-simplices according to the effective resistance of their boundaries approximately preserves the spectrum of the $(d-1)$-dimensional up-Laplacian [23].

1.1 Our Contributions

In this paper, we continue this trend of investigating effective resistance in simplicial complexes. Our main contribution is to show that there is a quantum algorithm for testing if a cycle is null-homologous in a simplicial complex that is parameterized by the effective resistance and effective capacitance of a cycle. The effective capacitance of a cycle is inspired by a quantity on graphs, and to our knowledge, we are the first to explore effective capacitance in simplicial complexes. Null-homology testing is an important primitive in computational topology. For example, the iterative algorithm for computing Betti numbers works by adding simplices one by one to the complex and testing if the boundary of each is already null-homolgous in the previous complex [4].

Motivated by our quantum algorithm, we investigate bounds on the effective resistance and effective capacitance. We prove a negative result. While the effective resistance between a pair of vertices in a graph is bounded above by the number of vertices in the graph, the effective resistance of a $(d-1)$-cycle in a simplicial complex can be exponential in the number of the d-simplices in the complex. The classical algorithm to determine if a cycle is null-homologous is to solve a system of linear equations in the dth boundary matrix, so the quantum algorithm is slower than the classical algorithm in the worst case. On the positive side, we prove this exponential behavior is the result of relative torsion in dimension $(d-1)$, and we prove bounds on effective resistance and effective capacitance in terms of the size of relative torsion groups of the complex.

We also prove a duality result between effective resistance and effective capacitance. For a d-dimensional simplicial complex $\mathcal{K}$ embedded in $\mathbb{R}^{d+1}$, the effective capacitance of certain d-cycles in $\mathcal{K}$ is the effective resistance between a pair of nodes in the dual graph. Our proof relies on a high dimensional generalization of planar graphs with two nodes s and t appearing on the same face. Due to space constraints, this result can be found in Appendix C.

2 Preliminaries

Given a set of vertices V, a **simplicial complex** $\mathcal{K}$ on V is a subset of the power set $\mathcal{K} \subseteq P(V)$ with the following property: for each $\tau \in \mathcal{K}$, if $\sigma \subset \tau$, then $\sigma \in \mathcal{K}$. We assume there is a fixed but arbitrary order on the vertices $V = (v_1, \ldots, v_n)$. A simplex $\sigma \in \mathcal{K}$ of size

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[1] Kook and Lee do not require the simplex to be in the complex.
[2] Hansen and Ghrist define effective resistance on a cellular sheaf, which is a generalization of a simplicial complex.

$|\sigma| = d + 1$ is a **d-simplex**. The set of all d-simplices of $\mathcal{K}$ is denoted $\mathcal{K}_d$. The **d-skeleton** of $\mathcal{K}$, denoted $\mathcal{K}^d$, is the simplicial complex of all simplices of $\mathcal{K}$ of size at most $d + 1$. The **dimension** of $\mathcal{K}$ is the largest d such that $\mathcal{K}$ contains a d-simplex; a 1-dimensional simplicial complex is a **graph**.

The **d$^{\text{th}}$ chain group** $C_d(\mathcal{K})$ is the vector space over $\mathbb{R}$ with orthonormal basis $\mathcal{K}_d$. Unless otherwise stated, all vectors and matrices will be in the basis $\mathcal{K}_d$. An element of $C_d(\mathcal{K})$ is a **d-chain**. Let $\sigma = \{v_{i_0}, \ldots, v_{i_d}\}$ be a d-simplex in $\mathcal{K}$ with $v_{i_j} \leq v_{i_k}$ whenever $j \leq k$. The **boundary** of σ is the $(d-1)$-chain $\partial\sigma = \sum_{j=0}^{d} (-1)^j (\sigma \setminus \{v_{i_j}\})$. The **d$^{\text{th}}$ boundary map** is the linear map $\partial_d : C_d(\mathcal{K}) \to C_{d-1}(\mathcal{K})$ defined $\partial_d f = \sum_{\sigma \in \mathcal{K}_d} f(\sigma)\partial\sigma$ where $f(\sigma)$ denotes the component of f indexed by the simplex σ. A key property of the boundary map is that $\partial_{d-1} \circ \partial_d = 0$. An element in $\ker \partial_d$ is a **cycle**, and an element in $\text{im} \, \partial_d$ is a **boundary** or a **null-homologous cycle**. The boundary maps have the property that $\partial_d \circ \partial_{d+1} = 0$, so $\text{im} \, \partial_{d+1} \subset \ker \partial_d$. The **d$^{\text{th}}$ homology group** is the quotient group $H_d(\mathcal{K}) = \ker(\partial_d) / \text{im}(\partial_{d+1})$. The **d$^{\text{th}}$ Betti number** β_d is the dimension of $H_d(\mathcal{K})$. The **d$^{\text{th}}$ coboundary map** is the map $\delta_d := \partial_{d+1}^T : C_d(\mathcal{K}) \to C_{d+1}(\mathcal{K})$. An element of $\ker \delta_d$ is a **cocycle**, and an element in $\text{im} \, \delta_{d-1}$ is a **coboundary**. We will use the notation $\partial[\mathcal{K}]$ and $\delta[\mathcal{K}]$ when we want to specify the complex associated with the (co)boundary operator. For some of our results we will need to consider integral homology. The **integral chain group** $C_d(\mathcal{K}, \mathbb{Z})$ is the free abelian group generated by the set $\mathcal{K}_d$ whose elements are formal sums $\sum_{\sigma_i \in \mathcal{K}_d} \alpha_i \sigma_i$ for $\alpha_i \in \mathbb{Z}$. The integral homology groups $H_d(\mathcal{K}, \mathbb{Z})$ are constructed in the same way as with coefficients over the reals.

The **d$^{\text{th}}$ up Laplacian**[3] is $L_d = \partial_{d+1}\delta_d$. There are two variants of the up Laplacian: the weighted up Laplacian and the normalized up Laplacian. Let $w : \mathcal{K}_{d+1} \to \mathbb{R}^+$ be a weight function on the $(d+1)$-simplices. Let $W : C_{d+1}(\mathcal{K}) \to C_{d+1}(\mathcal{K})$ be the diagonal matrix with $W_{\tau,\tau} = w(\tau)$. The **d$^{\text{th}}$ weighted up Laplacian** is $L_d^W = \partial_{d+1}W\delta_d$. The **degree** of a d-simplex σ is $\deg(\sigma) = \sum_{\tau \in \mathcal{K}_{d+1} : \, \sigma \subset \tau} w(\tau)$. Let $D : C_d(\mathcal{K}) \to C_d(\mathcal{K})$ be the diagonal matrix with $D_{\sigma,\sigma} = \deg(\sigma)$. The **d$^{\text{th}}$ normalized up Laplacian** is $\tilde{L}_d = D^{-1/2}\partial_{d+1}W\delta_d D^{-1/2}$.

We will make use of the **bra-ket notation** for vectors when discussing the quantum algorithm as this is the convention in the quantum computing literature. A **bra** is a row vector represented by the notation $\langle v|$. A **ket** is a column vector represented by the notation $|v\rangle$. The inner product of u and v is represented as $\langle u|v\rangle$.

Finally, we will often want to refer to the set of simplices given a non-zero value by a chain f. We call this set the **support** of f and denote it $\text{supp}(f) = \{\sigma_i \in \mathcal{K}_d : f(\sigma_i) \neq 0\}$.

3   Effective Resistance and Effective Capacitance

Let $\gamma \in C_{d-1}(\mathcal{K})$ be a cycle in a simplicial complex. We associate two quantities with γ, its *effective resistance* and *effective capacitance*. The effective resistance is finite if and only if γ is null-homologous, and the effective capacitance is finite if and only if γ is not null-homologous. We begin with the definition of effective resistance.

▶ **Definition 1.** *Let $\mathcal{K}$ be a simplicial complex with weight function $w : \mathcal{K}_d \to \mathbb{R}^+$. Let γ be a $(d-1)$-cycle in $\mathcal{K}$. If γ is null-homologous, the **effective resistance** of γ is $\mathcal{R}_\gamma(\mathcal{K}, W) = \gamma^T \left(L_d^W\right)^+ \gamma$, where $\left(L_d^W\right)^+$ is the Moore-Penrose pseudoinverse of L_d^W; if γ is not null-homologous, then $\mathcal{R}_\gamma(\mathcal{K}, W) = \infty$. When obvious, we drop the weights from the notation and write $\mathcal{R}_\gamma(\mathcal{K})$.*

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[3] There are related operators called the *down Laplacian* and the *Laplacian* (see [8]), but we won't use either in this paper.

This definition of effective resistance is consistent with effective resistance in graphs (see [26]) and other definitions of effective resistance in simplicial complexes [18, 23, 9]. However, this definition gives little intuition about effective resistance. We now prove there is an alternative definition of effective resistance in terms of chains with boundary γ. We begin with two definitions.

▶ **Definition 2.** *Given a d-dimensional simplicial complex $\mathcal{K}$ and a $(d-1)$-dimensional null-homologous cycle γ, a **unit γ-flow** is a d-chain $f \in C_d(\mathcal{K})$ such that $\partial f = \gamma$.*

In the case of graphs a unit γ-flow is a flow sending 1 unit of flow from its source to its sink. We now define the *flow energy* of a unit γ-flow, which quantifies the size of the flow.

▶ **Definition 3.** *Given a d-dimensional simplicial complex $\mathcal{K}$ with weight function $w : C_d(\mathcal{K}) \to \mathbb{R}^+$ and a unit γ-flow f, the **flow energy** of f on $\mathcal{K}$ is $\mathsf{J}(f) = \sum_{\sigma \in \mathcal{K}^{(d)}} \frac{f(\sigma)^2}{w(\sigma)} = f^T W^{-1} f$ where W is the diagonal matrix whose entries are the weights of the d-simplices.*

We will now relate unit γ-flows and their energy to the definition of effective resistance.

▶ **Lemma 4.** *Let $\mathcal{K}$ be a simplicial complex and let γ be a null-homologous d-cycle. The effective resistance of γ is the minimum flow energy over all unit γ-flows, i.e. $\mathcal{R}_\gamma(\mathcal{K}) = \min\{\mathsf{J}(f) \mid \partial f = \gamma\}$*

Proof. We use two well-known properties of the pseudoinverse.
1. If a matrix $B = AA^T$, then $B^+ = (A^T)^+ A^+$.
2. Let B be a matrix and $|v\rangle \in \operatorname{im} B$. The vector $B^+ v$ is the minimum-norm vector that B maps to v, i.e. $B^+ v = \arg\min\{\|u\| : Bu = v\}$.

Our first observation is that we can factor the weighted Laplacian as

$$L_d^W = \partial_{d+1} W \delta_d$$
$$= (\partial_{d+1} W^{1/2})(\partial_{d+1} W^{1/2})^T.$$

By property 1 above, $(L_d^W)^+ = ((\partial_{d+1} W^{1/2})^T)^+ (\partial_{d+1} W^{1/2})^+$. Therefore,

$$\mathcal{R}_\gamma(\mathcal{K}) = \gamma^T ((\partial_{d+1} W^{1/2})^T)^+ (\partial_{d+1} W^{1/2})^+ \gamma$$
$$= \|(\partial W^{1/2})^+ \gamma\|^2.$$

By property 2 above, $\mathcal{R}_\gamma(\mathcal{K})$ is the minimum squared-norm of a vector that $\partial_{d+1} W^{1/2}$ maps to γ. Let $f = (\partial W^{1/2})^+ \gamma$; the vector f is the unit γ-flow of minimum flow energy, which we now prove. A vector v is mapped to γ by $\partial W^{1/2}$ if and only if $W^{1/2} v$ is mapped to γ by ∂ as $W^{1/2}$ is a bijection, i.e. $W^{1/2} v$ is a unit γ-flow. Moreover, the flow energy of $W^{1/2} v$ is $\mathsf{J}(W^{1/2} v) = (W^{1/2} v)^T W^{-1} (W^{1/2} v) = v^T v = \|v\|^2$. Therefore, the minimum flow energy of a unit γ-flow is the minimum squared-norm of a vector that $\partial W^{1/2}$ maps to γ, which we previously saw was $\mathcal{R}_\gamma(\mathcal{K})$. ◀

We say that $(\partial W^{1/2})^+ \gamma$ is the **minimum-energy** unit γ-flow.

The definition of effective capacitance is less intuitive than the definition for effective resistance, both in graphs and simplicial complexes. As well, there are fewer results about effective capacitance in graphs than effective resistance and no previous results about effective capacitance in simplicial complexes. The effective capacitance of γ in a graph is the minimum energy of any *unit γ-potential*, which is analogous to effective resistance that was the minimum energy of any unit γ-flow. Before providing the definition of unit γ-potential in a simplicial complex we will begin by reviewing the definition of a unit *st*-potential in a graph, which can be found in [12].

Let G be a graph such that s and t are connected in G, and let $H \subseteq G$ be a subgraph such that s and t are not connected in H. A **unit st-potential** is a function $p \colon V(G) \to \mathbb{R}$ such that $p(t) = 1$, $p(s) = 0$, and for any two vertices u, v in the same connected component of H, $p(u) = p(v)$. Viewing p as a 0-chain we see by the last property that its coboundary is zero in H. Intuitively, our definition of a unit γ-potential measures "how far" a cycle γ is from null-homologous in a subcomplex $\mathcal{L}$ of $\mathcal{K}$. The definition of a unit γ-potential is analogous to the defintion of a unit st-potential.

▶ **Definition 5.** *Let $\mathcal{L} \subset \mathcal{K}$ be simplicial complexes, and let $\gamma \in C_{d-1}(\mathcal{L})$ be a $(d-1)$-cycle such that γ is null-homologous in $\mathcal{K}$ but not $\mathcal{L}$. A* **unit γ-potential** *in $\mathcal{L}$ is a $(d-1)$-chain p such that $\delta[\mathcal{L}]p = 0$ and $p^T \gamma = 1$.*

▶ **Definition 6.** *Given simplicial complexes $\mathcal{L} \subset \mathcal{K}$ with weight function $w \colon C_d(\mathcal{K}) \to \mathbb{R}$ and a γ-potential p in $\mathcal{L}$, the* **potential energy** *of p on $\mathcal{K}$ is $\mathcal{J}(p) = \sum_{\sigma \in \mathcal{K}_d}((\delta[\mathcal{K}]p)^T \sigma)^2 w(\sigma) = (\delta[\mathcal{K}]p)^T W(\delta[\mathcal{K}]p)$.*

Figure 1 in the appendix shows a γ-potential. It is not obvious from the definition that a unit γ-potential will even exist for γ. We prove this in the following lemma.

▶ **Lemma 7.** *Let $\mathcal{L} \subset \mathcal{K}$ be simplicial complexes whose $(d-1)$-skeletons are equal, and let $\gamma \in C_{d-1}(\mathcal{L})$ be a cycle. Then there exists a unit γ-potential in $\mathcal{L}$ if and only if γ is not null-homologous in $\mathcal{L}$.*

Proof. Observe that $\ker \delta_{d-1}[\mathcal{L}] = (\operatorname{im} \partial_d[\mathcal{L}])^\perp$ as $\delta_{d-1}[\mathcal{L}] = \partial[\mathcal{L}]^T$. Assume there is a γ-potential p in $\mathcal{L}$. As $\delta[\mathcal{L}]p = 0$, then $p \in \ker \delta_{d-1}[\mathcal{L}] = (\operatorname{im} \partial_d[\mathcal{L}])^\perp$. As $p^T \gamma = 1$ we see that γ has a non-zero component in $(\operatorname{im} \partial_d[\mathcal{L}])^\perp$, so $\gamma \notin \operatorname{im} \partial[\mathcal{L}]$. Alternatively, suppose that γ is not null-homologous in $\mathcal{L}$. Then γ has a non-zero component in $(\operatorname{im} \partial_d[\mathcal{L}])^\perp = \ker \delta[\mathcal{L}]$. Let $q = \Pi_{\ker \delta[\mathcal{L}]} \gamma$, where $\Pi_{\ker \delta[\mathcal{L}]}$ is the projection onto $\ker \delta[\mathcal{L}]$. Then $q^T \gamma \neq 0$ and $\delta[\mathcal{L}]q = 0$. The vector q is not necessarily a unit γ-potential as it is not necessarily the case that $q^T \gamma = 1$, but the scaled vector $p = \frac{1}{q^T \gamma}$ is a unit γ-potential.   ◀

Just as the effective resistance of γ was the minimum energy of any unit γ-flow, the effective capacitance of γ is the minimum energy of any unit γ-potential.

▶ **Definition 8.** *Let $\mathcal{L} \subset \mathcal{K}$ be simplicial complexes, and let $\gamma \in C_{d-1}(\mathcal{L})$ be a $(d-1)$-cycle that is null-homologous in $\mathcal{K}$. If γ is not null-homologous in $\mathcal{L}$, the* **effective capacitance** *of γ in $\mathcal{L}$ is $\mathcal{C}_\gamma(\mathcal{L}) = \min_p \mathcal{J}(p)$ where p is a γ-potential. If γ is null-homologous in $\mathcal{L}$, then $\mathcal{C}_\gamma(\mathcal{L}) = \infty$.*

4   Basic Properties: Parallel, Series, and Monotonicity Formulas

We now prove there are formulas for effective resistance in simplicial complexes analogous to the series and parallel formulas for effective resistance in graphs. These formulas not only are useful for calculating effective resistance, but they also provide intuition for effective resistance. In particular, they provide justification for the claim that effective resistance measures "how null-homologous" a cycle is in a complex.

▶ **Theorem 9** (Series Formula). *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be simplicial complexes with $\gamma \in C_{d-1}(\mathcal{K}_1) \cap C_{d-1}(\mathcal{K}_2)$, $C_d(\mathcal{K}_1) \cap C_d(\mathcal{K}_2) = \emptyset$, and γ null-homologous in $\mathcal{K}_1$ and $\mathcal{K}_2$. Let $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. Then $\mathcal{R}_\gamma(\mathcal{K}) \leq \mathcal{R}_{\gamma_1}(\mathcal{K}_1) + \mathcal{R}_{\gamma_2}(\mathcal{K}_2)$. Equality is achieved when γ_1 and γ_2 are the unique chains in $\mathcal{K}_1$ and $\mathcal{K}_2$ that sum to γ.*

Proof. Let γ_1 and γ_2 be null-homologous cycles in $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively that sum to γ, and let f_1 and f_2 be the minimum-energy unit γ_1- and γ_2-flows, respectively. Then $f_1 + f_2$ is a unit γ-flow, and we can bound $\mathcal{R}_\gamma(\mathcal{K}) \leq \mathsf{J}(f) = \mathsf{J}(f_1) + \mathsf{J}(f_2) = \mathcal{R}_{\gamma_1}(\mathcal{K}_1) + \mathcal{R}_{\gamma_2}(\mathcal{K}_2)$; the equality $\mathsf{J}(f) = \mathsf{J}(f_1) + \mathsf{J}(f_2)$ follows from the fact that $\mathcal{K}_1$ and $\mathcal{K}_2$ have disjoint sets of d-simplices.

To prove the other direction, observe that γ can always be written as the sum of two null-homologous chains $\gamma_1 \in C_{d-1}(\mathcal{K}_1)$ and $\gamma_2 \in C_{d-1}(\mathcal{K}_2)$. Any unit γ-flow g defines null-homologous $(d-1)$-cycles γ_1 and γ_2 that sum to γ; namely, if g_1 and g_2 are the restriction of g to $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, then $\gamma_1 = \partial g_1$ and $\gamma_2 = \partial g_2$.

If γ can be uniquely decomposed as $\gamma = \gamma_1 + \gamma_2$, then any unit γ-flow f can be decomposed as a unit γ_1-flow f_1 and a unit γ_2-flow f_2. It follows that the energy of f is minimized when the energy of f_1 and f_2 are both minimized. Hence, $\mathcal{R}_\gamma(\mathcal{K}) = \mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2)$.   ◄

▶ **Theorem 10** (Parallel Formula). *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be simplicial complexes with $\gamma \in C_{d-1}(\mathcal{K}_1) \cap C_{d-1}(\mathcal{K}_2)$, $C_d(\mathcal{K}_1) \cap C_d(\mathcal{K}_2) = \emptyset$, and γ null-homologous in $\mathcal{K}_1$ and $\mathcal{K}_2$. Let $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. Then $\mathcal{R}_\gamma(\mathcal{K}) \leq \left(\frac{1}{\mathcal{R}_\gamma(\mathcal{K}_1)} + \frac{1}{\mathcal{R}_\gamma(\mathcal{K}_2)} \right)^{-1}$. Equality is achieved when $\mathrm{im}\,\partial[\mathcal{K}_1] \cap \mathrm{im}\,\partial[\mathcal{K}_2] = \mathrm{span}\{\gamma\}$.*

Proof. Let f_1 and f_2 be the minimum energy unit γ-flows in $\mathcal{K}_1$ and $\mathcal{K}_2$ resp. For any real number t, the chain $g_t = t f_1 + (1-t) f_2$ is a unit γ-flow in $\mathcal{K}$. We can therefore bound the effective resistance as $\mathcal{R}_\gamma(\mathcal{K}) \leq \min_t \mathsf{J}(g_t)$.

To get the tightest bound of $\mathcal{R}_\gamma(\mathcal{K})$, we now derive $t_{\mathrm{opt}} := \arg\min_t \mathsf{J}(g_t)$. Observe that $\mathsf{J}(g_t) = t^2 \mathsf{J}(f_1) + (1-t)^2 \mathsf{J}(f_2) = t^2 \mathcal{R}_\gamma(\mathcal{K}_1) + (1-t)^2 \mathcal{R}_\gamma(\mathcal{K}_2)$; this follows from the fact that $\mathcal{K}_1$ and $\mathcal{K}_2$ have disjoint sets of d-simplices. The quantity $\mathsf{J}(g_t)$ is a positive quadratic in t, so t_{opt} is the value of t where the derivative of $\mathsf{J}(g_t)$ with respect to t is 0. Taking the derivative, we find that $t_{\mathrm{opt}} = \mathcal{R}_\gamma(\mathcal{K}_2)/(\mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2))$. Plugging t_{opt} into $\mathsf{J}(g_{t_{\mathrm{opt}}})$, we find that

$$
\begin{aligned}
\mathsf{J}(g_{t_{\mathrm{opt}}}) &= \left(\frac{\mathcal{R}_\gamma(\mathcal{K}_2)}{\mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2)} \right)^2 \mathcal{R}_\gamma(\mathcal{K}_1) + \left(1 - \frac{\mathcal{R}_\gamma(\mathcal{K}_2)}{\mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2)} \right)^2 \mathcal{R}_\gamma(\mathcal{K}_2) \\
&= \left(\frac{\mathcal{R}_\gamma(\mathcal{K}_2)}{\mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2)} \right)^2 \mathcal{R}_\gamma(\mathcal{K}_1) + \left(\frac{\mathcal{R}_\gamma(\mathcal{K}_1)}{\mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2)} \right)^2 \mathcal{R}_\gamma(\mathcal{K}_2) \\
&= \left(\frac{1}{\mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2)} \right)^2 \left(\mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2) \right) \mathcal{R}_\gamma(\mathcal{K}_1) \mathcal{R}_\gamma(\mathcal{K}_2) \\
&= \frac{\mathcal{R}_\gamma(\mathcal{K}_1) \mathcal{R}_\gamma(\mathcal{K}_2)}{\mathcal{R}_\gamma(\mathcal{K}_1) + \mathcal{R}_\gamma(\mathcal{K}_2)} \\
&= \left(\frac{1}{\mathcal{R}_\gamma(\mathcal{K}_1)} + \frac{1}{\mathcal{R}_\gamma(\mathcal{K}_2)} \right)^{-1}.
\end{aligned}
$$

This implies the upper bound on $\mathcal{R}_\gamma(\mathcal{K})$ in the theorem statement. To get the lower bound, observe that any unit γ-flow g in $\mathcal{K}$ can be orthogonally decomposed into chains $g_1 \in C_d(\mathcal{K}_1)$ and $g_2 \in C_d(\mathcal{K}_2)$, so $\partial[\mathcal{K}]g_1 + \partial[\mathcal{K}]g_2 = \gamma$. We claim that $\partial[\mathcal{K}_1]g_1 = t\gamma$ and $\partial[\mathcal{K}_2]g_2 = (1-t)\gamma$ for some value of t; if not, then $\partial[\mathcal{K}_1]g_1 = t\gamma + \eta$ and $\partial[\mathcal{K}_2]g_2 = (1-t)\gamma - \eta$ for some non-zero $\eta \notin \mathrm{span}\{\gamma\}$, which cannot be the case as $\mathrm{im}\,\partial[\mathcal{K}_1] \cap \mathrm{im}\,\partial[\mathcal{K}_2] = \mathrm{span}\{\gamma\}$. This proves the chain g is a linear combination of a unit γ-flow in $\mathcal{K}_1$ and a unit γ-flow in $\mathcal{K}_2$. The chain g_t is the lowest energy such linear combination.   ◄

Figure 2 in the appendix shows examples of unit γ-flows in series and parallel. These formulas justify the claim that the effective resistance of a null-homologous cycle γ is a measure of *how* null-homologous γ is. The more chains with boundary γ, the smaller the effective resistance of γ by the parallel formula. The smaller the chains bounding γ, the lower the effective resistance by the series formula.

A similar result to the series and parallel formula for effective resistance in graphs is *Rayleigh monotonicity*. Rayleigh monotonicity says that adding edges to a graph can only decrease the effective resistance between any pair of vertices; this reinforces the notion that effective resistance measures how well-connected a pair of vertices are, as adding an edge can only make a pair of vertices better connected. We prove a similar result for simplicial complexes.

▶ **Theorem 11** (Rayleigh Monotonicity). *Let $\mathcal{K} \subset \mathcal{L}$ be simplicial complexes. Let $|\gamma\rangle \in C_{d-1}(\mathcal{K}) \cap C_{d-1}(\mathcal{L})$ be a null-homologous cycle in both complexes. Then $\mathcal{R}_\gamma(\mathcal{L}) \leq \mathcal{R}_\gamma(\mathcal{K})$.*

Proof. As $C_d(\mathcal{K}) \subset C_d(\mathcal{L})$, then any unit γ-flow in $\mathcal{K}$ is also a unit γ-flow in $\mathcal{L}$. As the effective resistance is the minimum energy of a unit γ-flow, then clearly $\mathcal{R}_\gamma(\mathcal{L}) \leq \mathcal{R}_\gamma(\mathcal{K})$. ◀

5 Bounds on resistance and capacitance

In this section, we provide upper bounds on the resistance and capacitance of a cycle γ in an unweighted simplicial complex $\mathcal{K}$. Our upper bounds are polynomial in the number of d-simplices and the cardinality of the torsion subgroup of the relative homology groups. In particular, our bounds on resistance and capacitance are dependent on the maximum cardinality of the torsion subgroup of the relative homology group $H_{d-1}(\mathcal{L}, \mathcal{L}_0, \mathbb{Z})$, where $\mathcal{L} \subset \mathcal{K}$ is a d-dimensional subcomplex and $\mathcal{L}_0 \subset \mathcal{L}$ is a $(d-1)$-dimensional subcomplex. In the worst case, our upper bounds are exponential in the number of d-simplices. There exist simplicial complexes such that the torsion subgroup of $H_{d-1}(\mathcal{K}, \mathbb{Z})$ has cardinality n while $\mathcal{K}$ only has $O(\log^{1/d} n)$ vertices [22]. Note that such a complex contains at most $O(n^d)$ d-simplices.

In Theorem 17 we provide an example of a simplicial complex containing a cycle γ whose effective resistance is exponential in the number of simplices in the complex. It is important to reiterate that our bounds are in terms of the torsion of the relative homology groups. There exist simplicial complexes with no torsion in their homology groups but that do have torsion in their relative homology groups. An example of this is the Möbius strip. The Möbius strip has no torsion, but it has torsion relative to its boundary [5].

Our results rely on a change of basis on the boundary matrix called the *normal form* which reveals information about the torsion subgroup of $H_{d-1}(\mathcal{K}, \mathbb{Z})$. We state the normal form theorem below.

▶ **Theorem 12** (Munkres, Chapter 1 Section 11 [21]). *There are bases for $C_d(\mathcal{K}, \mathbb{Z})$ and $C_{d-1}(\mathcal{K}, \mathbb{Z})$ such that the matrix for the boundary operator $\partial_d \colon C_d(\mathcal{K}, \mathbb{Z}) \to C_{d-1}(\mathcal{K}, \mathbb{Z})$ is in **normal form**, i.e. $\tilde{\partial}_d = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where D is a diagonal matrix with entries $d_1, \ldots, d_m$ such that each d_i divides d_{i+1} and each 0 is a zero matrix of appropriate dimensionality. The normal form of ∂_d satisfies the following properties:*

1. *The entries $d_1, \ldots, d_m$ correspond to the torsion coefficients of $H_{d-1}(\mathcal{K}, \mathbb{Z}) \cong \mathbb{Z}^{\beta_d} \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m}$ (where $\mathbb{Z}_1 = 0$),*
2. *The number of zero columns is equal to the dimension of $\ker(\partial_d)$.*

Moreover, the boundary matrix ∂ in the standard basis can be transformed to $\tilde{\partial}$ by a set of elementary row and column operations. If ∂ is square, these operations multiply $\det \partial$ by ± 1.

Using Theorem 12, we obtain an upper bound on the determinants of the square sub-matrices of the boundary matrix $\partial_d[\mathcal{K}]$ in terms of the *relative homology groups* of $\mathcal{K}$. Let $\mathcal{L}$ be d-dimensional subcomplex of $\mathcal{K}$, and let $\mathcal{L}_0$ be a $(d-1)$-dimensional subcomplex of $\mathcal{K}$. The **relative boundary matrix** $\partial_d[\mathcal{L}, \mathcal{L}_0]$ is the submatrix of ∂_d obtained by including the columns of the d-simplices in $\mathcal{L}$ and excluding the rows of the $(d-1)$-simplices in $\mathcal{L}_0$. With the relative boundary matrices, one can define the **relative homology groups** as $H_k(\mathcal{L}, \mathcal{L}_0, \mathbb{Z}) = \ker \partial_k[\mathcal{L}, \mathcal{L}_0]/ \operatorname{im} \partial_{k+1}[\mathcal{L}, \mathcal{L}_0]$. More information on the relative boundary matrix can be found in [5]. We denote the cardinality of the torsion subgroup of the relative homology group $H_{d-1}(\mathcal{L}, \mathcal{L}_0, \mathbb{Z})$ by $\mathcal{T}(\mathcal{L}, \mathcal{L}_0)$. Similarly, we denote the maximum $\mathcal{T}(\mathcal{L}, \mathcal{L}_0)$ over all relative homology groups as $\mathcal{T}_{\max}(\mathcal{K})$.

▶ **Lemma 13.** *Let $\partial_d[\mathcal{L}, \mathcal{L}_0]$ be a $k \times k$ square submatrix of ∂_d constructed by including columns for the d-simplices $\mathcal{L}$ and excluding rows for the $(d-1)$-simplices $\mathcal{L}_0$. The magnitude of the determinant of $\partial_d[\mathcal{L}, \mathcal{L}_0]$ is bounded above by the cardinality of the torsion subgroup of $H_{d-1}(\mathcal{L}, \mathcal{L}_0, \mathbb{Z})$, i.e. $|\det (\partial_d[\mathcal{L}, \mathcal{L}_0])| \leq \mathcal{T}(\mathcal{L}, \mathcal{L}_0)$.*

Proof. Without loss of generality, we assume that $\det(\partial_d[\mathcal{L}, \mathcal{L}_0]) \neq 0$; if $\det(\partial_d[\mathcal{L}, \mathcal{L}_0]) = 0$, the bound is trivial. Since $\partial_d[\mathcal{L}, \mathcal{L}_0]$ is a non-singular square matrix, its normal form $\tilde{\partial}_d[\mathcal{L}, \mathcal{L}_0]$ is a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_k)$. By Theorem 12, the determinant of $\partial_d[\mathcal{L}, \mathcal{L}_0]$ is equal to $\pm \det(\tilde{\partial}_d[\mathcal{L}, \mathcal{L}_0]) = \prod_{i=1}^{k} d_i$. Also by Theorem 12, the torsion subgroup of $H_{d-1}(\mathcal{L}, \mathcal{L}_0)$ is $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ which has cardinality $\mathcal{T}(\mathcal{L}, \mathcal{L}_0) = \prod_{i=1}^{k} d_i$. ◀

We are now ready to upper bound the effective resistance of a cycle γ in a simplicial complex $\mathcal{K}$.

▶ **Theorem 14.** *Let $\mathcal{K}$ be an unweighted d-dimensional simplicial complex with n d-simplices. Let γ be a unit-length null-homologous $(d-1)$-cycle in $\mathcal{K}$. The effective resistance of γ is bounded above as $\mathcal{R}_\gamma(\mathcal{K}) = O\left(n^2 \cdot \mathcal{T}_{\max}(\mathcal{K})^2\right)$.*

Proof. First, we remove d-simplices from $\mathcal{K}$ to create a new complex $\mathcal{L}$ such that $\ker(\partial_d[\mathcal{L}]) = 0$ and $\operatorname{im} \partial_d[\mathcal{K}] = \operatorname{im} \partial_d[\mathcal{L}]$. Theorem 11 proves that removing d-simplices only increases the effective resistance, so $\mathcal{R}_\gamma(\mathcal{K}) \leq \mathcal{R}_\gamma(\mathcal{L})$. As $\ker(\partial_d[\mathcal{L}]) = 0$, there is a unique unit γ-flow $f \in C_{d-1}(\mathcal{L})$, which implies $\mathcal{R}_\gamma(\mathcal{L}) = ||f||^2$.

The matrix $\partial_d[\mathcal{L}]$ has full column rank, so we can find a non-singular $n_d \times n_d$ square submatrix of $\partial_d[\mathcal{L}]$; call this submatrix B. Let $\mathcal{L}_0$ be the $(d-1)$-dimensional subcomplex that contains the $(d-1)$-simplices corresponding to rows excluded from B; B is the relative boundary matrix $\partial_d[\mathcal{L}, \mathcal{L}_0]$. We have that $Bf = c$, where c is the restriction of γ to the rows of B. Observe that $\|c\| \leq \|\gamma\| = 1$

We will apply Cramer's rule to upper bound the size of f. Let $f(\sigma)$ denote the component of f indexed by the d-simplex σ. Cramer's rule gives the equality $f(\sigma) = \frac{\det(B_{\sigma,c})}{\det(B)}$ where $B_{\sigma,c}$ is the matrix obtained by replacing the column of B indexed by σ with the vector c. Since $\det(B)$ is integral, $|\det(B)| \geq 1$, we can drop the denominator and use the inequality $|f(\sigma)| \leq |\det(B_{\sigma,c})|$. We bound $|\det(B_{\sigma,c})|$ by its cofactor expansion on the column c, specifically $|\det(B_{\sigma,c})| = \left|\sum_{i=1}^{n_d}(-1)^{i+j} \cdot c_i \cdot \det(B_{\sigma,c}^{c,i})\right| \leq \sum_{i=1}^{n_d} |c_i| \cdot t(\mathcal{K}) = O\left(\sqrt{n} \cdot \mathcal{T}_{\max}(\mathcal{K})\right)$ where $B_{\sigma,c}^{c,i}$ denotes the submatrix obtained by removing the column c and removing the ith row and c_i denotes the ith component of c. The first inequality comes from Lemma 13, as $B_{\sigma,c}^{c,i}$ is the relative boundary matrix $\partial[\mathcal{L} \setminus \{\sigma\}, \mathcal{L}_0 \cup \sigma_i]$, where σ_i is the $(d-1)$-simplex corresponding to the ith row of B. The factor of $\sqrt{n}$ comes from our assumption that $\|c\| \leq 1$ and the fact that $\sum_{i=1}^{n_d} |c_i| = \|c\|_1 \leq \sqrt{|\operatorname{supp}(c)|} \cdot \|c\|_2 \leq \sqrt{n} \cdot \|c\|_2$, which can be shown using the Cauchy-Schwarz inequality. Finally, we compute the flow energy of f as $\mathsf{J}(f) = \sum_{\sigma \in \mathcal{K}^d} f(\sigma)^2 \leq \sum_{i=1}^{n_d} n \cdot \mathcal{T}_{\max}(\mathcal{K})^2 = O\left(n^2 \cdot \mathcal{T}_{\max}(\mathcal{K})^2\right)$. The effective resistance of γ is the flow energy of f, so the result follows. ◀

The same argument also applies for any subcomplex $\mathcal{L} \subset \mathcal{K}$ where γ is null-homologous in $\mathcal{L}$ which gives us the following corollary.

▶ **Corollary 15.** *Let $\mathcal{L} \subset \mathcal{K}$ be an unweighted d-dimensional simplicial complex and γ a null-homologous $(d-1)$-cycle in $\mathcal{L}$. The effective resistance of γ in $\mathcal{L}$ is bounded above by $\mathcal{R}_\gamma(\mathcal{L}) = O\left(n^2 \cdot \mathcal{T}_{\max}(\mathcal{K})^2\right)$.*

In order to upper bound the effective capacitance of γ, we need to make two additional assumptions. We need to assume that as input γ has integral coefficients and that each coefficient is bound above by a constant. Even though γ is given as input as an integral cycle, we still normalize γ as a preprocessing step. Normalizing γ does not change whether or not γ is null-homologous. We state the theorem but leave the proof to the appendix. The proof idea is similar to that of the bound on effective resistance, but with a few extra technical details.

▶ **Theorem 16.** *Let $\mathcal{L} \subset \mathcal{K}$ be an unweighted d-dimensional simplicial complexes, and let $\hat{\gamma} \in C_{d-1}(\mathcal{L})$ be a $(d-1)$-cycle that is null-homologous in $\mathcal{K}$ but not in $\mathcal{L}$. Assume also that $\hat{\gamma}$ is integral and each of the coefficients $|\hat{\gamma}_i| = O(1)$. The effective capacitance of $\gamma := \hat{\gamma}/\|\hat{\gamma}\|$ in $\mathcal{K}(x)$ is bounded above by $\mathcal{C}_\gamma(\mathcal{L}) = O\left(n^3 \cdot d \cdot \mathcal{T}_{\max}(\mathcal{K})^2\right)$.*

Recall that $t(\mathcal{K})$ could be exponential in the size of the complex. To end the section, we now provide an example of a simplicial complex containing a cycle γ such that the effective resistance of γ is exponential in the size of the complex.

▶ **Theorem 17.** *There exists a 2-dimensional simplicial complex with $\Theta(n)$ triangles and a cycle γ such that the effective resistance of γ is $\Theta(2^{2n})$.*

Proof. Let $\mathbb{RP}_\gamma$ denote a simplicial complex homeomorphic to the real projective plane with a disk removed; the cycle γ boundary of the removed disk. Hence, we have that the boundary of the sum of the triangles in the complex is $\partial_2 \mathbb{RP}_\gamma = 2\alpha + \gamma$ for some 1-cycle α. We require $\mathbb{RP}_\gamma$ to be triangulated in such a way that $|\operatorname{supp}(\alpha)| = |\operatorname{supp}(\gamma)|$; that is, α and γ contain the same number of edges. Let the constant c denote the number of triangles in $\mathbb{RP}_\gamma$. See Figure 3 in the appendix for a triangulation of $\mathbb{RP}_\gamma$.

We consider a collection of disjoint complexes $\mathbb{RP}_{\gamma_0}, \mathbb{RP}_{\gamma_1}, \dots, \mathbb{RP}_{\gamma_{n-1}}, \mathcal{D}_{\gamma_n}$. Each $\mathbb{RP}_{\gamma_i}$ is constructed in the same way was $\mathbb{RP}_\gamma$, and $\mathcal{D}_{\gamma_n}$ triangulation of a disk using c triangles with boundary γ_n such that $|\operatorname{supp}(\gamma_n)| = |\operatorname{supp}(\gamma)|$. The sum of the triangles of each $\mathbb{RP}_{\gamma_i}$ has boundary $\partial_2 \mathbb{RP}_{\gamma_i} = 2\alpha_i + \gamma_i$.

We consider the simplicial complex $\mathcal{K}$ constructed by taking the quotient space under the identification $\alpha_i \sim \gamma_{i+1}$. That is, we glue the cycle α_i in $\mathbb{RP}_{\gamma_i}$ along the cycle γ_{i+1} in $\mathbb{RP}_{\gamma_{i+1}}$. The resulting complex contains a unique unit γ_0-flow f. The chain f assigns a value of 1 to each triangle in $\mathbb{RP}_{\gamma_0}$, a value of -2 to each triangle in $\mathbb{RP}_{\gamma_1}$, and in general, a value of $(-1)^i 2^{i+1}$ to each triangle in $\mathbb{RP}_{\gamma_i}$ and D_{γ_n}. To see that f is indeed a unit γ_0-flow we compute the boundary

$$\partial f = \left(\sum_{i=0}^{n-1} (-1)^i \cdot 2^i \partial \mathbb{RP}_{\gamma_i}\right) + (-1)^n \cdot 2^n \partial \mathcal{D}_{\gamma_n}$$

$$= \left(\sum_{i=0}^{n-1} (-1)^i \cdot 2^i (\gamma_i + 2\gamma_{i+1})\right) + (-1)^n \cdot 2^n \gamma_n$$

$$= \gamma_0.$$

The chain f is unique because the value of f must be equal to 1 on each triangle in $\mathbb{RP}_{\gamma_0}$, and the values on the triangles of $\mathbb{RP}_{\gamma_i}$ determines the values on the triangles of $\mathbb{RP}_{\gamma_{i+1}}$ and $\mathcal{D}_{\gamma_n}$ as the γ_i terms must cancel out in ∂f.

As f is the unique unit γ_0-flow, the effective resistance of f is the flow energy of γ. The flow energy of f is $\mathsf{J}(f) = \sum_{i=0}^{n} c \cdot (2^i)^2 = \frac{c}{3}\left(2^{2n+2} - 1\right) = \Theta(2^{2n})$. ◄

6   A Quantum Algorithm for Null-Homology Testing

In this section we provide an application of effective resistance and capacitance in simplicial complexes. We show that a quantum algorithm based on the *span program* model can be used to decide whether or not a cycle γ is null-homologous in a simplicial complex $\mathcal{K}$. An introduction to the span program model can be found in Appendix 6.1. The query complexity of this algorithm is parameterized by the maximum (finite) effective resistance and capacitance of γ over all subcomplexes of $\mathcal{K}$.

Our algorithm is a generalization of the quantum algorithm developed by Belovs and Reichardt to decide st-connectivity in a graph [1]. Their algorithm is parameterized by the effective resistance and capacitance of the 0-cycle $|t\rangle - |s\rangle$ in the graph. Upper bounds on the effective resistance and capacitance imply a query complexity of $O(n^{3/2})$, where n is the number of vertices [12].

Our upper bounds on effective resistance and capacitance imply that the query complexity is polynomial in both the number of d-simplices as well as the cardinality of the largest torsion subgroup of a relative homology group of $\mathcal{K}$. In the case that $\mathcal{K}$ is a graph, we match the $O(n^{3/2})$ upper bound. Under the assumptions that $\mathcal{K}$ is relative torsion free and that γ is the boundary of a d-simplex (which may or may not be included in the complex) we also match the $O(n^{3/2})$ upper bound. Note that these assumptions are always true for st-connectivity in graphs.

6.1   A brief introduction to span programs

Span programs were first defined by Karchmer and Wigderson [14] and were first used for quantum algorithms by Reichardt and Špalek [24]. Intuitively, a span program is a model of computation which encodes a boolean function $f\colon \{0,1\}^n \to \{0,1\}$ into the geometry of two vector spaces and a linear operator between them. Encoding f into a span program implies the existence of a quantum query algorithm evaluating f (Theorem 20.)

▶ **Definition 18.** *A **span program** $\mathcal{P} = (\mathcal{H}, \mathcal{U}, |\tau\rangle, A)$ over the set of strings $\{0,1\}^n$ is a 4-tuple consisting of:*

1. *A finite dimensional Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ where $\mathcal{H}_i = \mathcal{H}_{i,0} \oplus \mathcal{H}_{i,1}$,*

2. *a vector space $\mathcal{U}$,*

3. *a non-zero vector $|\tau\rangle \in \mathcal{U}$, called the **target vector***

4. *a linear operator $A\colon \mathcal{H} \to \mathcal{U}$.*

For every string $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ we associate the Hilbert space $\mathcal{H}(x) = \mathcal{H}_{1,x_1} \oplus \cdots \oplus \mathcal{H}_{N,x_n}$ and the linear operator $A(x) = A\Pi_{\mathcal{H}(x)}\colon \mathcal{H} \to \mathcal{U}$ where $\Pi_{\mathcal{H}(x)}$ is the projection of $\mathcal{H}$ onto $\mathcal{H}(x)$.

The quantum query complexity of evaluating $\mathcal{P}$ depends on the sizes of the positive and negative witnesses, which we now define.

▶ **Definition 19.** *Let $\mathcal{P}$ be a span program and let $x \in \{0,1\}^N$. A **positive witness** for x is a vector $|w\rangle \in \mathcal{H}(x)$ such that $A|w\rangle = |\tau\rangle$. The **positive witness size** of x is*

$$w_+(x, \mathcal{P}) = \min\{\||w\rangle\|^2 : |w\rangle \in \mathcal{H}(x),\ A|w\rangle = |\tau\rangle\}.$$

*If no positive witness exists for x, then $w_+(x, \mathcal{P}) = \infty$. If there is a positive witness for x, then x is a **positive instance**.*

*A **negative witness** for x is a linear map $\langle w| : \mathcal{U} \to \mathbb{R}$ such that $\langle w|A\Pi_{\mathcal{H}(x)} = 0$ and $\langle w|\tau\rangle = 1$. The **negative witness size** of x is*

$$w_-(x, \mathcal{P}) = \min\{\|\langle w|A\|^2 : \langle w| : \mathcal{U} \to \mathbb{R},\ \langle w|A\Pi_{\mathcal{H}(x)} = 0,\ \langle w|\tau\rangle = 1\}.$$

*If no negative witness exists for x, then $w_-(x, \mathcal{P}) = \infty$. If there is a negative witness for x, then x is a **negative instance**.*

A string $x \in \{0,1\}^N$ will either be a positive or negative instance of $\mathcal{P}$. A span program $\mathcal{P}$ **decides** the function $f: \{0,1\}^n \to \{0,1\}$ if $f(x) = 1$ when x is a positive instance and $f(x) = 0$ when x is a negative instance. A span program can also evaluate a partial boolean function $g: D \to \{0,1\}$ where $D \subset \{0,1\}^n$ by the same criteria.

Span programs are a popular method in quantum computing because there are upper bounds on the complexity of evaluating span programs in the **query model**. The query model evaluates the complexity of a quantum algorithm by its **query complexity**, the number of times it queries an input oracle. In our case, the input oracle returns the bits of the binary string x. The **input oracle** $\mathcal{O}_x$ takes $\mathcal{O}_x : |i\rangle|b\rangle \to |i\rangle|b \oplus x_i\rangle$ where $i \in [N]$. Observe that the states $|i\rangle$ can be stored on $\lceil \log N \rceil$ qubits. Reichardt [25] showed that the query complexity of a span program is a function of the positive and negative witness sizes of the program.

▶ **Theorem 20** (Reichardt [25]). *Let $D \subset \{0,1\}^N$ and $f : D \to \{0,1\}$. Let $\mathcal{P}$ be a span program that decides f. Let $W_+(f, \mathcal{P}) = \max_{x \in f^{-1}(1)} w_+(x, \mathcal{P})$ and $W(f, \mathcal{P})_- = \max_{x \in f^{-1}(0)} w_-(x, \mathcal{P})$. There is a bounded error quantum algorithm that decides f with query complexity $O\left(\sqrt{W_+(f,\mathcal{P})W_-(f,\mathcal{P})}\right)$.*

A caveat to the query complexity model is that in general the time complexity of an algorithm can be much larger than the query complexity. We will provide details on bounding the time complexity of our problem in Section A.

6.2 A span program to decide if a cycle is null-homologous

In this section we present a span program for testing if a cycle is null-homologous in a simplicial complex. This span program is a generalization of the span program for *st*-connectivity defined in [14] and used to develop quantum algorithms in [1, 2, 12, 13]. Let $\mathcal{K}$ be a d-dimensional simplicial complex. Let $|\gamma\rangle \in C_{d-1}(\mathcal{K})$ be a $(d-1)$-cycle. Let n be the number of d-simplices in $\mathcal{K}$. Order the d-simplices $\{\sigma_1, \ldots, \sigma_n\}$. Let $w: \mathcal{K}_d \to \mathbb{R}$ be a weight function on the d-simplices. We define a span program over the strings $\{0,1\}^n$ in the following way.
1. $\mathcal{H} = C_d(\mathcal{K})$, with $\mathcal{H}_{i,1} = \text{span}\{|\sigma_i\rangle\}$ and $\mathcal{H}_{i,0} = \{0\}$.
2. $\mathcal{U} = C_{d-1}(\mathcal{K})$
3. $A = \partial_d\sqrt{W}: C_d(\mathcal{K}) \to C_{d-1}(\mathcal{K})$
4. $|\tau\rangle = \gamma$

We denote the above span program by $\mathcal{P}_{\mathcal{K}}$. Let $x \in \{0,1\}^N$ be a binary string. We define the subcomplex $\mathcal{K}(x) := \mathcal{K}^{d-1} \cup \{\sigma_i : x_i = 1\}$. That is, $\mathcal{K}(x)$ contains the d-simplices σ_i such that $x_i = 1$. There exists a solution to the linear system $\partial_d \sqrt{W} \Pi_{\mathcal{K}(x)} |v\rangle = |\gamma\rangle$ if and only if the cycle $|\gamma\rangle$ is null-homologous in $\mathcal{K}(x)$ if and only if x is a positive instance of $\mathcal{P}_{\mathcal{K}}$. The span program $\mathcal{P}_{\mathcal{K}}$ decides the boolean function $f : \{0,1\}^n \to \{0,1\}$ where $f(x) = 1$ if and only if γ is a null-homologous cycle in the subcomplex $\mathcal{K}(x)$.

Given a string $x \in \{0,1\}^n$ we show in the following two lemmas that $w_+(x, \mathcal{P}_{\mathcal{K}}) = \mathcal{R}_\gamma(\mathcal{K}(x))$ and $w_-(x, \mathcal{P}_{\mathcal{K}}) = \mathcal{C}_\gamma(\mathcal{K}(x))$. The proofs are simple calculations following from the definitions of effective resistance and capacitance.

▶ **Lemma 21.** *Let $x \in \{0,1\}^N$ be a positive instance. There is a bijection between positive witnesses $|w\rangle$ for x and unit γ-flows $|f\rangle$ in $\mathcal{K}(x)$. Moreover, the positive witness size is equal to the effective resistance of γ in $\mathcal{K}(x)$; that is, $w_+(x, \mathcal{P}_{\mathcal{K}}) = \mathcal{R}_\gamma(\mathcal{K}(x))$.*

Proof. Let $|w\rangle \in C_d(\mathcal{K})$ be a positive witness for x, so $\partial_d \sqrt{W} |w\rangle = |\gamma\rangle$. We construct a unit γ-flow $|f\rangle$ in $\mathcal{K}(x)$ by $f|=\rangle \sqrt{W}|w\rangle$; $|f\rangle$ is indeed a unit γ-flow as $\partial_d |f\rangle = \partial_d \sqrt{W} |w\rangle = |\gamma\rangle$. Moreover, $|w\rangle = W^{-1/2}|f\rangle$. The flow energy of γ is $\mathsf{J}(f) = \langle f|W^{-1}|f\rangle = \langle W^{-1/2}f|W^{-1/2}f\rangle = \langle w|w\rangle = \||w\rangle\|^2$. Hence, the flow energy of $|f\rangle$ equals the witness size of $|w\rangle$. Conversely, let $|f\rangle$ be a unit γ-flow in $\mathcal{K}(x)$ and define the positive witness for x as $|w\rangle = W^{-1/2}|f\rangle$. The same computation in the above paragraph shows that the flow energy of f equals the positive witness size of $|w\rangle$. ◀

▶ **Lemma 22.** *Let $x \in \{0,1\}^N$ be a negative instance. There is a bijection between negative witnesses $\langle w_-|$ for x and unit γ-potentials $\langle p|$ in $\mathcal{K}(x)$. Moreover, the negative witness size is equal to the effective capacitance of γ in $\mathcal{K}(x)$; that is, $w_-(x, \mathcal{P}_{\mathcal{K}}) = C_\gamma(\mathcal{K}(x))$.*

Proof. Let $\langle w|$ be a negative witness for x. We will verify that $\langle w|$ is a unit γ-potential. We have by the definition of negative witness that $\langle w|\gamma\rangle = 1$. We must show that the coboundary of $\langle w|$ is zero in $\mathcal{K}(x)$. By the definition of a negative witness we have $\langle w|\partial_d \sqrt{W} \Pi_{\mathcal{K}(x)} = 0$ Since $\sqrt{W}$ is a diagonal matrix and $\Pi_{\mathcal{K}(x)}$ restricts the coboundary to the subcomplex $\mathcal{K}(x)$, we see that $\langle w|\partial_d|\sigma\rangle = 0$ for any $\sigma \in \mathcal{K}(x)_d$. To show that the witness size of $\langle w|$ is equal to the potential energy, we have

$$\|\langle w|\partial_d \sqrt{W}\|^2 = \langle w|\partial_d \sqrt{W} \sqrt{W} \partial_d^T |w\rangle$$
$$= \langle \delta_{d-1} w|W|\delta_{d-1} w\rangle$$
$$= \mathcal{J}(w).$$

Conversely, let $\langle p|$ be a unit γ-potential for $\mathcal{K}(x)$. We will prove that $\langle p|$ is a negative witness for γ. Since the coboundary of $\langle p|$ is zero in $\mathcal{K}(x)$ we have $\langle \delta p|\sigma\rangle = 0$ for each $\sigma \in \mathcal{K}(x)_d$, which implies $\langle p|\partial_d \sqrt{W} \Pi_{\mathcal{K}(x)} = 0$ by the reasoning in the previous paragraph. Also by the previous paragraph, we have that the potential energy of $\langle p|$ is equal to the negative witness size of $\langle p|$ which concludes the proof. ◀

From these two lemmas we obtain the main theorem of the section, the quantum query complexity of the span program.

▶ **Theorem 23.** *Given a d-dimensional simplicial complex $\mathcal{K}$, a $(d-1)$-dimensional cycle γ that is null-homologous in $\mathcal{K}$, and a d-dimensional subcomplex $\mathcal{K}(x) \subseteq \mathcal{K}$, there exists a quantum algorithm deciding whether or not γ is null-homologous in $\mathcal{K}(x)$ whose quantum query complexity is $O\left(\sqrt{\mathcal{R}_{\max}(\gamma)\mathcal{C}_{\max}(\gamma)}\right)$, where $\mathcal{R}_{\max}$ is the maximum effective resistance of γ in any subcomplex $\mathcal{K}(y)$ and $\mathcal{C}_{\max}$ is the maximum effective capacitance γ in any subcomplex $\mathcal{K}(y)$.*

Proof. By Theorem 20, the span program $\mathcal{P}_{\mathcal{K}}$ can be converted into a quantum algorithm whose query complexity is $O\left(\sqrt{W_+(f,\mathcal{P}_{\mathcal{K}})W_-(f,\mathcal{P}_{\mathcal{K}})}\right)$ where $W_+(f,\mathcal{P}_{\mathcal{K}}) = \max_{x \in f^{-1}(1)} \mathcal{R}_\gamma(\mathcal{K}(x)) = \mathcal{R}_{\max}(\gamma)$ and $W_-(f,\mathcal{P}_{\mathcal{K}}) = \max_{x \in f^{-1}(0)} \mathcal{C}_\gamma(\mathcal{K}(x)) = \mathcal{C}_{\max}(\gamma)$. ◀

By Theorems 14 and 16 we obtain an upper bound on the query complexity parameterized by the number of simplices and the cardinality of the torsion subgroups of the relative homology groups.

▶ **Theorem 24.** *Let $\mathcal{K}$ be an unweighted d-dimensional simplicial complex and $\mathcal{K}(x)$ a d-dimensional subcomplex $\mathcal{K}(x) \subseteq \mathcal{K}$. Let $\hat{\gamma} \in C_{d-1}(\mathcal{K})$ be a $(d-1)$-cycle such that $\hat{\gamma}$ is integral and each of the coefficients $|\hat{\gamma}_i| = O(1)$. There exists a quantum algorithm deciding whether or not $\gamma := \hat{\gamma}/\|\hat{\gamma}\|$ is null-homologous in $\mathcal{K}(x)$ whose quantum query complexity is $O\left(n^{5/2} \cdot d^{1/2} \cdot \mathcal{T}_{\max}(\mathcal{K})^2\right)$.*

Finally, we state the query complexity under some assumptions that arise in the case of st-connectivity in graphs. In this case, the input cycle is $|t\rangle - |s\rangle$, and the support of $|t\rangle - |s\rangle$ is equal to 2. A factor of n in both the upper bounds on resistance and capacitance is actually a factor of $|\operatorname{supp}(\gamma)|$ as seen in the proofs of these bounds. Under the assumption that the support of γ is bounded above by $O(d)$, we can replace a factor of n from both the flow energy and potential energy of any unit γ-flow and unit γ-potential with a factor of d. This assumption on the size of $\operatorname{supp}(\gamma)$ is true when γ is the boundary of a d-simplex.

Furthermore, graphs do not contain relative torsion[4], so we make the additional assumption that $\mathcal{K}$ is relative torsion-free. Under these assumptions our query complexity matches the query complexity arising from the span program deciding st-connectivity.

▶ **Corollary 25.** *Let $\mathcal{K}$ be an unweighted d-dimensional simplicial complex and $\mathcal{K}(x) \subseteq \mathcal{K}$ be a d-dimensional subcomplex. Let $\hat{\gamma} \in C_{d-1}(\mathcal{K})$ be a $(d-1)$-cycle such that $\hat{\gamma}$ is integral and each of the coefficients $|\hat{\gamma}_i| = O(1)$. Further assume that $\mathcal{K}$ is relative torsion-free and $|\operatorname{supp}(\gamma)| = O(d)$. There exists a quantum algorithm deciding whether or not $\gamma := \hat{\gamma}/\|\hat{\gamma}\|$ is null-homologous in $\mathcal{K}(x)$ whose quantum query complexity is $O\left((dn)^{3/2}\right)$.*

6.3 Time efficient implementations

We have given bounds on the query complexity of null-homology testing; however, this does not imply a bound on the time complexity of evaluating this span program. There are two obstacles to a time-efficient implementation of the span program: the weights and the input cycle γ. The weights on the d-simplices make it difficult to implement the matrix $\partial\sqrt{W}$, as the weights on the simplices can be arbitrary real numbers. The input cycle γ is difficult to create on a quantum computer for the same reason, as the entries of γ can also be arbitrary real numbers. We explore the implementation details of this algorithm in the appendix.

We can give a quantum algorithm of bounded time complexity in one particular instance: when $\mathcal{K}$ is unweighted and γ is the boundary of a d-simplex. We do not require said d-simplex to actually appear in the complex. The time complexity of this case is given in the following theorem.

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[4] Graphs do not have relative torsion as the boundary matrix of a graph $\partial_1[G]$ is *totally unimodular*. See Section 5.1 of the paper [5] for an explanation of the relationship between totally unimodularity of the boundary matrix and relative torsion of a simplicial complex.

▶ **Theorem 26.** *Let $\mathcal{K}$ be a simplicial complex, $\gamma \in C_{d-1}(\mathcal{K})$ a null-homologous cycle, and $\mathcal{K}(x) \subset \mathcal{K}$ be a simplicial complex. Furthermore, assume that γ is the boundary of a d-simplex and the complex is unweighted. There is a quantum algorithm for deciding if γ is null-homologous in $\mathcal{K}(x)$ that runs in time $\tilde{O}\left(\frac{(dn)^{3/2}\cdot\mathcal{T}_{\max}(\mathcal{K})^2}{\sqrt{\lambda}}\left(\sqrt{d}+\sqrt{d_{\max}}\right)+\left(\sqrt{\frac{1}{\mathcal{R}_\gamma(\mathcal{K})}}+\sqrt{\mathcal{R}_\gamma(\mathcal{K})}\right)\left(\sqrt{d}+\sqrt{d_{\max}}\right)\right)$ where $d_{\max}$ is the maximum degree of a $(d-1)$-simplex in $\mathcal{K}$ and λ is the smallest eigenvalue of the normalized up-Laplacian.*

The factor of $\tilde{O}\left(\left(\sqrt{\frac{1}{\mathcal{R}_\gamma(\mathcal{K})}}+\sqrt{\mathcal{R}_\gamma(\mathcal{K})}\right)+\left(\sqrt{d}+\sqrt{d_{\max}}\right)\right)$ comes from the time necessary to create the initial state to the algorithm. Assuming such a state is already provided the running time reduces to $\tilde{O}\left(\frac{(dn)^{3/2}\cdot\mathcal{T}_{\max}(\mathcal{K})^2}{\sqrt{\lambda}}\left(\sqrt{d}+\sqrt{d_{\max}}\right)\right)$.

We can get a tighter anaylsis when $\mathcal{K}$ is a pseudomanifold. When $\mathcal{K}$ is a pseudomanifold, the maximum degree $d_{\max} \leq 2$ and the running time reduces to $\tilde{O}\left(\frac{d^2\cdot n^{3/2}\cdot\mathcal{T}_{\max}(\mathcal{K})^2}{\sqrt{\lambda}}\right)$. Steenbergen, Klivans, and Mukherjee provide a lower bound on λ for pseudomanifolds similar to the Cheeger inequality for graphs [28]. Their lower bound is in terms of the **boundary expansion** of the complex which is defined as $h_d := \min_{\substack{\phi\in C_d(\mathbb{Z}_2) \\ \phi\notin\mathrm{im}\,\partial_{d+1}}} \frac{\|\partial_d\phi\|}{\min_{\phi\in\mathrm{im}\,\partial_{d+1}}\|\phi+\psi\|}$. For pseudomanifolds they prove a lower bound of $\lambda \geq \frac{h_d^2}{2(d+1)}$.

Finally, in the case of graphs there is no relative torsion and we can state the running time as $\tilde{O}\left(\frac{n^{3/2}}{\sqrt{\lambda}}\right)$ where λ is bounded below by the well-known Cheeger inequality.

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A   Evaluating the span program for null-homology

In this section, we give a quantum algorithm for evaluating the null-homology span program. Our algorithm is inspired by the quantum algorithm for evaluating st-connectivity span program in graphs. The first quantum algorithm for evaluating the st-connectivity span program was given by Belovsz and Reichardt in [1]; however, we follow the slightly different algorithm introduced by Ito and Jefferies in [11]. We are also greatly indebted to the presentation of this algorithm given by Jeffery and Kimmel in [13].

The algorithm for evaluating a general span program $\mathcal{P} = (\mathcal{H}, \mathcal{U}, |\tau\rangle, A)$ is to perform phase estimation of the vector $|w_0\rangle := A^+|\tau\rangle$ on the unitary operator $U = R_{\mathcal{H}(x)} R_{\ker A}$ where the notation R_S denotes the reflection about the subspace S. (The unitary $R_S = 2\Pi_S - I$, where Π_S is the projection onto S.) Intuitively, if x is a positive instance, then $|w_0\rangle$ will be close to an eigenvector of U with phase 0. If x is a negative instance, then $|w_0\rangle$ will be far from any eigenvector of U of phase 0. If we want to evaluate the function $f : D \to \{0, 1\}$, we need to perform phase estimation to precision $O\left(1/\sqrt{W_-(f, \mathcal{P})W_+(f, \mathcal{P})}\right)$. The algorithm for phase estimation of a unitary U to precision $O(\delta)$ performs $O(1/\delta)$ implementations of the unitary U [16], so the algorithm for evaluating the span program $\mathcal{P} = (\mathcal{H}, \mathcal{U}, |\tau\rangle, A)$ requires $O\left(\sqrt{W_-(f, \mathcal{P})W_+(f, \mathcal{P})}\right)$ implementations of U.

We now analyze the time complexity of implementing the unitary U. The reflection $R_{\mathcal{H}(X)}$ can be implemented with one query to $\mathcal{O}_x$. This reflection is the same as the reflection across the good states in Grover's Algorithm. The rest of this section is devoted to an implementation of $R_{\ker \partial}$.

Recall that $\ker \partial_d \subset C_d(\mathcal{K})$. The idea behind the implementation of $R_{\ker \partial}$ is that instead of reflecting across $\ker \partial_d$ directly, we can embed $C_d(\mathcal{K})$ into $C_{d-1}(\mathcal{K}) \otimes C_d(\mathcal{K})$ by sending $|\tau\rangle \to c|\partial\tau\rangle|\tau\rangle$ (where c is a normalization constant). We can then implement the reflection $R_{\ker \partial}$ by implementing a series of "local reflections" on the basis $|\partial\tau\rangle|\tau\rangle$.

We consider two subspaces B and C of $C_{d-1}(K) \otimes C_d(K)$. The spaces B and C are defined:

$$B = \text{span}\left\{|b_\tau\rangle := \frac{1}{\sqrt{d+1}}|\partial\tau\rangle|\tau\rangle : \tau \in \mathcal{K}_d\right\}$$

and

$$C = \text{span}\left\{|c_\sigma\rangle := \sum_{\sigma \subset \tau} \sqrt{\frac{w(\sigma)}{\deg(\sigma)}}|\sigma\rangle|\tau\rangle : \sigma \in \mathcal{K}_{d-1}\right\}.$$

The space $C_{d-1}(\mathcal{K}) \otimes C_d(\mathcal{K})$ has basis $\{|\sigma\rangle|\tau\rangle \mid \sigma \in K_{d-1}, \tau \in \mathcal{K}_d\}$. The vector $|b_\tau\rangle$ is non-zero on a basis element $|\sigma\rangle|\tau\rangle$ if and only if σ is on the boundary of τ. Similarly, a component of $|c_\sigma\rangle$ is non-zero on $|\sigma\rangle|\tau\rangle$ if and only if τ is on the coboundary of σ. The vector $|b_\tau\rangle$ can be thought of as being *like* the boundary of τ, with the additional property that the set $\{|b_\tau\rangle \mid \tau \in C_d(\mathcal{K})\}$ is orthonormal. Similarly, the vector $|c_\sigma\rangle$ is *like* the coboundary of σ but orthonormal.

We also define operators that embed $C_d(\mathcal{K})$ and $C_{d-1}(\mathcal{K})$ into B and C respectively. We define linear operators $M_B \colon C_d(\mathcal{K}) \to B$ and $M_C \colon C_{d-1}(\mathcal{K}) \to C$ as follows:

$$M_B := \sum_{\tau \in \mathcal{K}_d} |b_\tau\rangle\langle\tau|,$$

and

$$M_C := \sum_{\sigma \in \mathcal{K}_{d-1}} |c_\sigma\rangle\langle\sigma|.$$

As the columns of M_B and M_C are orthonormal, both operators are isometries.

We introduce the matrices M_C and M_B as they have the property that $\ker M_C^\dagger M_B = \ker \partial$, which we prove in the follow lemma. This fact will give us a way to implement $R_{\ker \partial}$.

▶ **Lemma 27.** $\ker M_C^\dagger M_B = \ker \partial$.

Proof. We first calculate the matrix $M_C^\dagger M_B$. We then argue that $\ker M_C^\dagger M_B = \ker \partial$. For a $(d-1)$-simplex σ and a d-simplex τ, we have that

$$\langle c_\sigma | b_\tau \rangle = \sum_{\tau' \in \mathcal{K}_d : \sigma \subset \tau'} \frac{w(\tau)}{\sqrt{\deg(\sigma)}} \langle \sigma | \partial \tau \rangle \langle \tau' | \tau \rangle = \begin{cases} \sqrt{\frac{w(\tau)}{(d+1)\deg(\sigma)}} \langle \sigma | \partial \tau \rangle & \text{if } \sigma \subset \tau \\ 0 & \text{otherwise} \end{cases}.$$

So $\langle c_\sigma | b_\tau \rangle$ is non-zero if and only if σ is in the boundary of τ. We use this to calculate the product $M_C^\dagger M_B$:

$$M_C^\dagger M_B = \sum_{\sigma \in \mathcal{K}_{d-1}} \sum_{\tau \in \mathcal{K}_d} |\sigma\rangle \langle c_\sigma | b_\tau \rangle \langle \tau |$$

$$= \frac{1}{\sqrt{(d+1)}} \sum_{\sigma \subset \tau} \frac{\sqrt{w(\tau)} \langle \sigma | \partial \tau \rangle}{\sqrt{\deg(\sigma)}} |\sigma\rangle \langle \tau |$$

$$= \frac{1}{\sqrt{(d+1)}} \left(\sum_{\sigma \in \mathcal{K}_{d-1}(\mathcal{K})} \frac{|\sigma\rangle \langle \sigma|}{\sqrt{\deg(\sigma)}} \right) \sum_{\tau \in \mathcal{K}_d} \sqrt{w(\tau)} |\partial \tau\rangle \langle \tau |$$

$$= \frac{1}{\sqrt{(d+1)}} \left(\sum_{\sigma \in \mathcal{K}_{d-1}(\mathcal{K})} \frac{|\sigma\rangle \langle \sigma|}{\sqrt{\deg(\sigma)}} \right) \partial \sqrt{W} =: \hat{\partial}.$$

The term $\frac{|\sigma\rangle \langle \sigma|}{\sqrt{\deg(\sigma)}}$ is all-zeros matrix except for the (σ, σ)-entry, which is $\frac{1}{\sqrt{\deg(\sigma)}}$. The sum $\sum_{\sigma \in \mathcal{K}_{d-1}} \frac{|\sigma\rangle \langle \sigma|}{\sqrt{\deg(\sigma)}}$ is a diagonal matrix. Accordingly, the matrix $\hat{\partial}$ is $\partial \sqrt{W}$ with each row scaled. Scaling the rows of a matrix does not change its row space or kernel, so $\ker M_C^\dagger M_B = \ker \partial$.   ◀

The spaces B and C and the matrices M_B and M_C are inspired by the follow lemma of Szegedy which is necessary for implementing $R_{\ker \partial}$.

▶ **Lemma 28** (Szegedy [29], Theorem 1). *Let M_B and M_C be matrices with the same number of rows and orthonormal columns, and let $B = \operatorname{span} M_B$ and $C = \operatorname{span} M_C$. The matrix $M_C^\dagger M_B$ has singular values at most 1. Let $\cos \theta_1, \ldots, \cos \theta_k$ be the singular values of $M_C^\dagger M_B$ in the range $(0, 1)$. Let $U = R_C R_B$. We can decompose the eigenspaces of U as*
- *The $(+1)$-eigenspace of U is $(B \cap C) \oplus (B^\perp \cap C^\perp)$.*
- *The (-1)-eigenspace of U is $(B \cap C^\perp) \oplus (B^\perp \cap C)$.*
- *The remaining eigenvalues of U are $e^{\pm 2i\theta_j}$ for $1 \leq j \leq k$.*

The following lemma gives us a way to implment the $R_{\ker \partial}$. Let R_{U^-} be the rotation about (-1)-eigenspace of U, and let $V = M_B^\dagger R_{U^-} M_B$. The matrix V embeds $C_d(\mathcal{K})$ into B with M_B, performs a reflection on B about the (-1)-eigenspace of U, and unembeds with $M_B^\dagger$. The following lemma proves that $V = R_{\ker \partial}$.

▶ **Lemma 29.** *The matrix $V = M_B^\dagger R_{U^-} M_B$ satisfies the equality $V = R_{\ker \partial}$.*

Proof. We first verify that V is a reflection; that is, we show the eigenvalues of V are 1 and -1. The matrices M_B and M_C have orthonormal columns, so we can use Lemma 28 to characterize the eigenspaces of U. The (-1)-eigenspace of U is $(B \cap C^\perp) \oplus (B^\perp \cap C)$ and the $(+1)$-eigenspace of U is $(B \cap C) \cap (B^\perp \cap C^\perp)$. As the spaces $(B \cap C)$ and $(B \cap C^\perp)$ span B,

then R_{U^-} restricted to B has eigenvalues 1 and -1. As $B = \operatorname{im} M_B$ and $V = M_B^\dagger R_{U^-} M_B$, then we conclude that V has eigenvalues 1 and -1 as well.

Now that we have determined that V is a reflection, we need to determine which subspace V reflects across. A corollary of the previous paragraph is that a vector $|\psi\rangle \in C_d(\mathcal{K})$ is in the $(+1)$-eigenspace of V if and only if $M_B|\psi\rangle$ is in the (-1)-eigenspace of U. Specifically, a vector $|\psi\rangle$ is in the $(+1)$-eigenspace of V if and only if $M_B|\psi\rangle \in C^\perp$. As $C^\perp = \ker M_C^\dagger$, the vector $|\psi\rangle$ is in the $(+1)$-eigenspace of V if and only if $|\psi\rangle \in \ker M_C^\dagger M_B$. We proved in Lemma 27 that $\ker M_C^\dagger M_B = \ker \partial$, so we conclude that $V = R_{\ker \partial}$. ◀

We have a matrix V that implements $R_{\ker \partial}$; next, we analyze the complexity of implementing V. We start by analyzing the complexity of implementing R_{U^-}, the reflection across the (-1)-eigenspace of U.

We implement the reflection around the (-1)-eigenspace of U using phase estimation, an algorithm introduced by Magniez et al. [19]. The algorithm is as follows. We first estimate the phase of U to some degree of accuracy to be specified shortly. Intuitively, we need to estimate the phase of U to high enough accuracy to distinguish between -1 eigenvalues of U and eigenvalues of U close to -1. We then perform a reflection controlled on the estimated phase.

The **_phase gap_** of a unitary U with eigenvalues $\{e^{i\theta_1}, \ldots, e^{i\theta_k}\}$ is $\min\{|\theta_i| : \theta_i \neq 0\}$. The following lemma shows that the phase gap determines the complexity of reflecting across the 1-eigenspace of U.

▶ **Lemma 30** (Magniez et al. [19], Paraphrase of Theorem 6). *Let U be a unitary with phase gap θ. A reflection around the 1-eigenspace of U can be performed to constant precision with $O\left(\frac{1}{\theta}\right)$ applications of U.*

The phase gap measures gap between the 1-eigenspace of a unitary and all other eigenvalues. We are interested in the gap in phase between the (-1)-eigenspace of U and the other eigenvalues of U. This is precisely the phase gap of $-U$. The following lemma analyzes the phase gap of $-U$ and gives the complexity of reflecting about the (-1)-eigenspace of U.

▶ **Lemma 31.** *We can implement R_{U^-} with $O\left(\sqrt{\frac{d+1}{\lambda}}\right)$ calls to U, where λ is the smallest non-zero eigenvalue of the normalized up-Laplacian.*

Proof. We need to calculate the phase gap of $-U$ to determine the precision to which we need to estimate the phase of U. Observe that if θ is the phase of an eigenvalue of U, then $\theta + \pi$ is the phase of an eigenvalue of $-U$. We can bound the phase gap of $-U$ using Lemma 28. The non-zero eigenvalues of U are $\{e^{\pm i2\theta_j}\}_j$, where $\{\cos\theta_j\}_j$ were the singular values of $M_C^\dagger M_B$. Therefore, the phases of $-U$ are $\{\pm|\pi - 2\theta_j|\}_j$. Using the inequality that $\pi/2 - \theta_j \geq \cos\theta_j$ for $\theta_j \in [0, \pi/2]$, then the phase gap of $-U$ is bounded below by

$$|\pi - 2\theta_j| \geq 2\cos\theta_j \geq 2 \cdot \sigma_{\min}(M_C^\dagger M_B)$$

where $\sigma_{\min}(M_C^\dagger M_B)$ is the smallest singular value of $M_C^\dagger M_B$.

We can actually relate the smallest singular value of $M_C^\dagger M_B$ to something more meaningful. By the proof of Lemma 27, the matrix $M_C^\dagger M_B = \frac{1}{\sqrt{d+1}} D^{-1/2} \partial \sqrt{W}$, where D is the diagonal matrix with the degrees of the $(d-1)$-simplices on the diagonal. Thus, $(M_C^\dagger M_B)(M_C^\dagger M_B)^\dagger = \frac{1}{d+1} D^{-1/2} \partial W \delta D^{-1/2} = \frac{1}{d+1} D^{-1/2} L D^{-1/2}$. Recall from Section 2 that the matrix $D^{-1/2} L D^{-1/2}$ is the normalied up-Laplacian. The singular values of a matrix A are the square roots of the eigenvalues of AA^T. Thus, the smallest singular value of $M_C^\dagger M_B$,

and the phase gap of $-U$, is $\Omega\left(\sqrt{\frac{\lambda}{d+1}}\right)$, where λ is the smallest eigenvalue of Δ. Therefore, by Lemma 30, we can implement R_{U^-} with $O\left(\sqrt{\frac{d+1}{\lambda}}\right)$ calls to U. ◄

We are almost ready to give the running time for $V = R_{\ker\partial}$, but first, we need to make a delicate distinction. The matrices M_B and M_C have orthonormal columns, but they are *not* unitary. We can see this as $\ker M_B^\dagger \neq 0$ and $\ker M_C^\dagger \neq 0$. As M_B and M_C are not unitary, they cannot be implemented on a quantum computer. Fortunately, it suffices to implment unitaries U_B and U_C such that $U_B|_{C_d(\mathcal{K})} = M_B$ and $U_C|_{C_{d-1}(\mathcal{K})} = M_C$. Now we can give the running time for $V = R_{\ker\partial}$.

▶ **Lemma 32.** *There is an algorithm to perform $R_{\ker\partial}$ in time $\tilde{O}\left(\sqrt{\frac{d+1}{\lambda}}(T_B + T_C)\right)$, where T_B and T_C are the times to perform U_B and U_C respectively.*

Proof. Lemma 29 shows that $R_{\ker\partial} = V = M_B^\dagger R_{U^-} M_B$. We can equivalently run $U_B^\dagger R_{U^-} U_B$. As U_B takes T_B by definition, we only need to show we can implement R_{U^-} in $\tilde{O}\left(\sqrt{(d+1)/\lambda}\,(T_B + T_C)\right)$ time. Lemma 31 shows we can implement R_{U^-} with $O(\sqrt{(d+1)/\lambda})$ calls to U, so we need to show we can implment U in $\tilde{O}(T_B + T_C)$. The unitary $U = R_C R_B$, and we claim we can implement R_C and R_B in $\tilde{O}(T_B)$ and $\tilde{O}(T_C)$ respectively. We can implement R_B as $U_B R_{\mathcal{K}_d} U_B^\dagger$, where $R_{\mathcal{K}_d}$ reflects across the basis states $\{|0\rangle|\sigma\rangle \mid \sigma \in \mathcal{K}_d\}$. We can check if a quantum state is of the form $|0\rangle|\sigma\rangle$ in $O(\log n_d)$ gates (specifically, by checking if the basis state is within a certain range), so the unitary $R_{\mathcal{K}_d}$ takes $O(\log n_d)$ gates, and R_B takes $\tilde{O}(T_B)$ time. The unitary R_C takes $\tilde{O}(T_C)$ time by the same argument. ◄

The running time T_B is dependent on how the boundary maps are loaded into the quantum algorithm. We propose a method of storing the boundary maps in a quantum computer called the **incidence array**. The incidence array is adapted from the *adjacency array* introduced by Durr et al. in [7] to store the adjancy between pairs of vertices in a graph.

For a d-simplex $\tau = \{v_0, \ldots, v_d\}$, the **down-incidence array** is the function $g : |\tau\rangle|j\rangle|0\rangle \to |\tau\rangle|j\rangle|\tau \setminus \{v_j\}\rangle$ for $0 \leq j \leq d$. The simplices in the boundary of τ have alternating sign. To address this, we also perform a negation conditioned on the parity of $|j\rangle$ to compute $(-1)^j|\tau\rangle|j\rangle|\tau \setminus \{v_j\}\rangle$.

Durr et al. [7] claim that queries to the incidence array can be performed in logarithmic time. As the down-incidence array is identical to the adjacency array[5], queries to the down-incidence also take logarithmic time. We can compute the state $|\partial\tau\rangle|\tau\rangle$ with the down-incidence array and the following lemma.

▶ **Lemma 33** (Cade, Montanaro, Belovs [2], Implicit in the proof of Lemma 2). *Let $f : [m] \to [k]$ be a function, and let $\mathcal{O}_f$ be an oracle that computes $\mathcal{O}_f : |i\rangle|0\rangle \to |i\rangle|f(i)\rangle$. The state $\frac{1}{\sqrt{m}}\sum_{i=1}^{m}|f(i)\rangle$ can be computed with $O(\sqrt{m})$ queries to $\mathcal{O}_f$ and $O(\mathrm{polylog}(m))$ additional gates.*

▶ **Corollary 34.** *The unitary U_B can be implemented in $O(\sqrt{d})$ queries to the down-incidence array and $\tilde{O}(\sqrt{d})$ time.*

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[5] The down-incidence array is actually an adjacency array of a graph related to simplicial complexes, namely, the incidence graph between the $(d-1)$- and d-simplices.

It is harder to produce a generic implementation of U_C than U_B. The d-simplices can have arbitrary weights, so constructing the states $|c_\sigma\rangle$ in general requires constructing arbitrary quantum states with real coefficients. However, the weights on the simplices do not affect whether or not a cycle is null-homologous. Therefore, we can always run our null-homology test on the unweighted complex; the trade-off is that the effective resistance or effective capacitance might be higher in the unweighted complex. We analyze the running time of this case in Section A.

We now analyze the complexity of constructing the initial state $|w_0\rangle/|||w_0\rangle||$. To construct the dummy state, we start by adding an additional "d-cell" $|\emptyset\rangle$ to the complex with boundary $|\gamma\rangle$ (really, we just add $|\gamma\rangle$ as a column to ∂.) The new cell will have non-trivial overlap with $|w_0\rangle$, so we can construct $|w_0\rangle$ by amplifying this component of $|\emptyset\rangle$. We outline this method in the proof of Theorem 36, but first, we state Lemma 35 which is a generalization of the parallel formula for effective resistance; its proof is nearly identical to the proof of Theorem 10.

▶ **Lemma 35.** *Let $V = V_1 \oplus V_2$ be a vector space. Let $A : V \to U$ be a linear map, and let $A_1 : U_1 \to V$ and $A_2 : U_2 \to V$ be the restriction of A to U_1 and U_2. Let $|t\rangle \in \operatorname{im} A_1 \cap \operatorname{im} A_2 \subset U$. If $|s\rangle = A^+|t\rangle$, $|s_1\rangle = A_1^+|t\rangle$, and $|s_2\rangle = A_2^+|t\rangle$, then*

$$||s||^2 \leq \left(\frac{1}{||s_1||^2} + \frac{1}{||s_2||^2} \right)^{-1}$$

Equality is achieved when $\operatorname{im} A_1 \cap \operatorname{im} A_2 = \operatorname{span}\{|t\rangle\}$. In this case, $|s\rangle = t|s_1\rangle + (1-t)|s_2\rangle$ where $t = ||s_2||^2/(||s_1||^2 + ||s_2||^2)$.

▶ **Theorem 36.** *Let $\mathcal{O}_\gamma$ be the oracle that takes $\mathcal{O}_\gamma : |0\rangle \to |\gamma\rangle$. Let T_γ be the time it takes to implement $\mathcal{O}_\gamma$. The state $|w_0\rangle = \partial^+|\gamma\rangle$ can be created in $\tilde{O}((\sqrt{1/\mathcal{R}_\gamma(\mathcal{K})} + \sqrt{\mathcal{R}_\gamma(\mathcal{K})})(T_B + T_C + T_\gamma))$ time.*

Proof. We append $|\gamma\rangle$ as a column to ∂ to create a new matrix $\hat{\partial}$. Let $|\emptyset\rangle$ be index of the new column, so $\hat{\partial} = \partial + |\gamma\rangle\langle\emptyset|$. Let $|w_0'\rangle = \hat{\partial}^+|\gamma\rangle$. We conclude that $|\emptyset\rangle = |w_0'\rangle + |w_0'^\perp\rangle$ where $|w_0'^\perp\rangle \in \ker\hat{\partial}$, as the projection $\Pi_{\ker\hat{\partial}^\perp}|\emptyset\rangle = \hat{\partial}^+\hat{\partial}|\emptyset\rangle = \hat{\partial}^+|\gamma\rangle = |\hat{w}_0\rangle$.

We construct $|w_0\rangle/|||w_0\rangle||$ in two steps. First, we use amplitude amplification to amplify the $|w_0'\rangle$ component of $|\emptyset\rangle$. We then use a second amplitude amplification to amplify the $|w_0\rangle$ component of $|w_0'\rangle$. These amplitude amplifications are nested, as we need to perform the first to create the initial state for the second.

If we perform constant time phase estimation of $|\emptyset\rangle$ on the unitary $R_{\ker\hat{\partial}}$, then we can map $|\emptyset\rangle$ to $|0\rangle|w_0'\rangle + |1\rangle|w_0^\perp\rangle$. We can then amplify the amplitude of $|0\rangle|w_0\rangle$ part arbitrarily close to $|w_0\rangle/|||w_0\rangle||$ using $O(|||w_0\rangle||^{-1})$ calls to $R_{\ker\hat{\partial}}$.

We calculate $|||w_0\rangle||$ using the formula from the lemma. The vector $|\emptyset\rangle$ has length 1, so Lemma 35 shows that

$$|||w_0'\rangle||^2 = \left(\frac{1}{\mathcal{R}_\gamma(\mathcal{K})} + 1 \right)^{-1} = \frac{\mathcal{R}_\gamma(\mathcal{K})}{\mathcal{R}_\gamma(\mathcal{K}) + 1}.$$

Thus, we need to perform the reflection $R_{\ker\hat{\partial}}$ a total of $O(|||w_0\rangle||^{-1}) = O(\sqrt{(\mathcal{R}_\gamma(\mathcal{K}) + 1)/\mathcal{R}_\gamma(\mathcal{K})})$ times to create $|\hat{w}_0\rangle/|||\hat{w}_0\rangle||$.

The next step in our algorithm is to amplify the $|w_0\rangle/\||w_0\rangle\|$ component of $|w_0'\rangle/\||w_0'\rangle\|$. By Lemma 35, the state $\||w_0'\rangle\| = t\||w_0\rangle\| + (1-t)\||\emptyset\rangle\|$ for $t = 1/(\mathcal{R}_\gamma(\mathcal{K}) + 1)$. Therefore, the $|w_0\rangle/\||w_0\rangle\|$ component of $|w_0'\rangle/\||w_0'\rangle\|$ has norm

$$t\||w_0\rangle\|/\||w_0'\rangle\| = \frac{1}{\mathcal{R}_\gamma(\mathcal{K}) + 1}\sqrt{\mathcal{R}_\gamma(\mathcal{K})}\sqrt{\frac{\mathcal{R}_\gamma(\mathcal{K}) + 1}{\mathcal{R}_\gamma(\mathcal{K})}}$$
$$= \sqrt{\frac{1}{\mathcal{R}_\gamma(\mathcal{K}) + 1}}$$

To return the state $|w_0\rangle/\||w_0\rangle\|$, we need to perform amplitude amplification again. We can create the state $|w_0'\rangle$ using the amplitude amplification from the previous two paragraphs with $O(\sqrt{(\mathcal{R}_\gamma(\mathcal{K}) + 1)/\mathcal{R}_\gamma(\mathcal{K})})$ applications of $R_{\ker \partial'}$, and we can reflect across $|\emptyset\rangle$ in constant time as it is a basis state. To create $|w_0\rangle/\||w_0\rangle\|$, we need

$$O\left(\sqrt{(\mathcal{R}_\gamma(\mathcal{K}) + 1)/\mathcal{R}_\gamma(\mathcal{K})}\sqrt{(\mathcal{R}_\gamma(\mathcal{K}) + 1)}\right) = O\left(\sqrt{\mathcal{R}_\gamma(\mathcal{K})} + \sqrt{\frac{1}{\mathcal{R}_\gamma(\mathcal{K})}}\right)$$

applications of $R_{\ker \hat\partial}$.

We now argue that we can compute $R_{\ker \hat\partial}$ in $O(T_C + T_B + T_\gamma)$ time. As was the case with $R_{\ker \partial}$, we decompose $R_{\ker \hat\partial} = M_{\hat B}^\dagger R_{\hat U^-} M_{\hat B}$ for space $\hat B$ and $\hat C$ defined

$$\hat B = B \cup \{|b_\emptyset\rangle := |\gamma\rangle\}$$

$$\hat C = \text{span}\left\{|c_\sigma\rangle := \frac{1}{\sqrt{\deg(\sigma) + 1}}|\sigma\rangle|\emptyset\rangle + \sum_{\sigma \subset \tau} \frac{w(\sigma)}{\sqrt{\deg(\sigma) + 1}}|\sigma\rangle|\tau\rangle : \sigma \in \mathcal{K}_{d-1}\right\}.$$

The unitaries $M_{\hat B}$ $M_{\hat C}$, and $\mathbb{R}_{\hat U^-}$ are defined analogously to M_B and M_C. We can implement the unitary version of these matrices $U_{\hat B}$ in $O(T_B + T_\gamma)$ and $U_{\hat C}$ in $O(T_C)$. ◀

We summarize this section in the following theorem.

▶ **Theorem 37.** *Let $\mathcal{K}$ be a simplicial complex, $\gamma \in C_{d-1}(\mathcal{K})$ a null-homologous cycle, and $\mathcal{K}(x) \subset \mathcal{K}$ be a simplicial complex. There is a quantum algorithm for deciding if γ is null-homologous in $\mathcal{K}(x)$ that runs in time*

$$\tilde{O}\left(\sqrt{\frac{(d+1)\mathcal{R}_{\max}(\mathcal{K})\mathcal{C}_{\max}(\mathcal{K})}{\lambda}}(\sqrt{d} + T_C) + \left(\sqrt{\frac{1}{\mathcal{R}_\gamma(\mathcal{K})}} + \sqrt{\mathcal{R}_\gamma(\mathcal{K})}\right)(\sqrt{d} + T_C + T_\gamma)\right)$$

Special cases

We now consider a few special cases of the null-homology span program. These special cases will allow us to replace the terms T_B and T_γ in Theorem 37 with concrete running times.

Unweighted simplicial complexes

We now consider the case where there are no weights on the d-simplices, or equivalently, when $w(\tau) = 1$ for each d-simplex τ. While computing M_C is hard in general, in the unweighted case, we can implement the unitary M_C using a straightforward oracle. For a $(d-1)$-simplex σ that is incident to the d-simplices $\{\tau_1, \ldots, \tau_m\}$, the **up-incidence array** is the oracle is the function that maps $h : |\sigma\rangle|j\rangle|0\rangle \to |\sigma\rangle|j\rangle|\tau_j\rangle$. By Lemma 33, the up-incidence array can be used to compute $|c_\tau\rangle = \frac{1}{\sqrt{m}}|\tau\rangle|\delta\tau\rangle$ in $O(\sqrt{m})$ time. The unitary M_C computes the state $|c_\sigma\rangle$ in parallel, so computing M_C will take $\sqrt{d_{\max}}$ queries, where $d_{\max} = \max_{\tau \in C_{d-1}(\mathcal{K})} \deg(\tau)$ time. This is summarized in the following lemma.

▶ **Lemma 38.** *If $\mathcal{K}$ is an unweighted simplicial complex, the unitary U_C can be implemented in $O(\sqrt{d_{\max}})$ queries to the up-incidence array and $\tilde{O}(\sqrt{d_{\max}})$ time.*

Cycle is the boundary of a d-simplex

We now consider the case that the input cycle γ is the boundary of a d-simplex. In this case, we can implement the oracle $\mathcal{O}_\gamma$ with the down incidence array used to implement M_B. We get the same running time for T_γ as T_B.

▶ **Lemma 39.** *If γ is the boundary of a d-simplex, there is a quantum algorithm implementing $\mathcal{O}_\gamma$ in $O(\sqrt{d})$ queries to the down incidence array and $\tilde{O}(\sqrt{d})$ time.*

B   Omitted proofs

Proof of Theorem 16

Before obtaining our upper bound on the effective capacitance of a cycle we need to prove one lemma. In the following lemma, we provide an upper bound on the largest singular value of the coboundary matrix.

▶ **Lemma 40.** *The largest singular value of the coboundary matrix δ_{d-1} is $\sigma_{max}(\delta_{d-1}) = O(\sqrt{dn})$.*

Proof. Recall that the $(d-1)$ up-Laplacian is $L = \delta_{d-1}^T \delta_{d-1}$. The squared singular values of δ_{d-1} are the eigenvalues of L; this follows from the generic theorem that the squared singular values of a matrix A are the eigenvalues of $A^T A$. Thus, $\sigma_{\max}(\delta_{d-1})^2 \leq \sum_i \sigma_i(\delta_{d-1})^2 = \text{trace}(L)$, where the $\sigma_i(\delta_{d-1})$ are the singular values of δ_{d-1}. We can obtain an upper bound on $\sigma_{\max}(\delta_{d-1})$ by computing the trace of L. The diagonal elements of L are the degrees of the $(d-1)$-simplices [8, Proposition 3.3.2]. Each d-simplex is the coface of $d+1$ $(d-1)$-simplices, so summing up the diagonal of L, we find $\text{trace}(L) = O(dn)$. Thus, $\sigma_{\max}(\delta_{d-1}$ is $O(\sqrt{dn})$.   ◄

▶ **Theorem 16.** *Let $\mathcal{L} \subset \mathcal{K}$ be an unweighted d-dimensional simplicial complexes, and let $\hat{\gamma} \in C_{d-1}(\mathcal{L})$ be a $(d-1)$-cycle that is null-homologous in $\mathcal{K}$ but not in $\mathcal{L}$. Assume also that $\hat{\gamma}$ is integral and each of the coefficients $|\hat{\gamma}_i| = O(1)$. The effective capacitance of $\gamma := \hat{\gamma}/\|\hat{\gamma}\|$ in $\mathcal{K}(x)$ is bounded above by $\mathcal{C}_\gamma(\mathcal{L}) = O\left(n^3 \cdot d \cdot \mathcal{T}_{\max}(\mathcal{K})^2\right)$.*

Proof. Observe that because the coefficients $\hat{\gamma}_i = O(1)$ then $\|\hat{\gamma}\| = O(\sqrt{n})$. We will use this fact later in the proof. Let p be a γ-potential. By definition, $\delta[\mathcal{L}]p = 0$ and $\gamma^T p = 1$. We can express these constraints as the linear system

$$
\begin{bmatrix} \delta[\mathcal{L}] \\ \\ \gamma^T \end{bmatrix} p = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
$$

We first remove linearly-dependent columns from this linear system until this system has full column rank. Columns of the matrix correspond to $(d-1)$ simplices of $\mathcal{L}$, and rows correspond to d-simplices of $\mathcal{L}$. Removing columns from $\delta[\mathcal{L}]$ changes $\delta[\mathcal{L}]$ to the relative coboundary matrix $\delta[\mathcal{L}, \mathcal{L}_0]$ where $\mathcal{L}_0$ is the $(d-1)$-subcomplex corresponding to the columns that were removed.

Removing linearly-dependent columns does not change the image of the system of equation, so there is still a solution r, i.e.

$$\begin{bmatrix} \delta[\mathcal{L}, \mathcal{L}_0] \\ \\ c \end{bmatrix} r = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where c is the subvector of γ^T after removing the columns. The vector r is not a γ-potential as it is a vector in $C_{d-1}(\mathcal{L} \setminus \mathcal{L}_0)$, not $C_{d-1}(\mathcal{L})$. However, we can extend r to be a γ-potential by adding zeros in the entries indexed by $\mathcal{L}_0$. Adding zero-valued entries preserves the length of r.

We now want to remove rows from this matrix so that it has full row rank. Topologically, removing rows corresponds to removing d-simplices from the complex $\mathcal{L}$ to create a new complex $\mathcal{L}_1$. Note that we must always include the row c to have full row rank; otherwise, r would be a non-zero vector in the kernel of this system, meaning the system does not have full rank. Removing these rows gives the linear system

$$\begin{bmatrix} \delta[\mathcal{L}_1, \mathcal{L}_0] \\ \\ c \end{bmatrix} r = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Let $C = \begin{bmatrix} \delta[\mathcal{L}_1, \mathcal{L}_0]^T & c^T \end{bmatrix}^T$ and $b = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$. Note that C is an square matrix of size (say) $m \times m$.

We now use Cramer's rule to bound the size of $\|r\|$. By Cramer's rule, r_i, the ith entry of r, is $r_i = \frac{\det(C_{i,b})}{\det(C)}$. where $C_{i,b}$ is the matrix obtained by replacing the ith column with b. We first lower bound $|\det(C)|$. We can express $\det(C)$ by its cofactor expansion on the row of c as $\det(C) = \sum_{i=1}^{m}(-1)^i \cdot c_i \cdot \det(\delta[\mathcal{L}_1, \mathcal{L}_0]_i)$ where $\delta[\mathcal{L}_1, \mathcal{L}_0]_i$ is $\delta[\mathcal{L}_1, \mathcal{L}_0]$ without the ith column. Each term $\delta[\mathcal{L}_1, \mathcal{L}_0]_i$ is integral as $\delta[\mathcal{L}_1, \mathcal{L}_0]$ is an integral matrix. Moreover, each term $c_i = \hat{\gamma}_{j_i}/\|\hat{\gamma}\|$ where $\hat{\gamma}_{j_i}$ is an integer, as c is a subvector of γ. We can then derive the lower bound

$$\begin{aligned} |\det(C)| &= \left| \sum_{i=1}^{m}(-1)^i \cdot c_i \cdot \det(\delta[\mathcal{L}_1, \mathcal{L}_0]_i) \right| \\ &= \frac{1}{\|\hat{\gamma}\|} \left| \sum_{i=1}^{m}(-1)^i \cdot \hat{\gamma}_{j_i} \cdot \det(\delta[\mathcal{L}_1, \mathcal{L}_0]_i) \right| \\ &\geq \frac{1}{\|\hat{\gamma}\|} && \text{(as } \sum_{i=1}^{m}(-1)^i \cdot \hat{\gamma}_{j_i} \cdot \det(\delta[\mathcal{L}_1, \mathcal{L}_0]_i) \text{ is integral)} \\ &= \Omega\left(\frac{1}{\sqrt{|\operatorname{supp}(\hat{\gamma})|}} \right) && \text{(as each entry } \hat{\gamma}_i \text{ is } O(1)) \\ &= \Omega\left(\frac{1}{\sqrt{n}} \right). \end{aligned}$$

We now upper bound $|\det(C_{i,b})|$. We calculate $\det(C_{i,b})$ with the cofactor expansion on the column replaced by b. As b has 1 in its last entry and 0s elsewhere, the cofactor expansion is $\det(C_{i,b}) = \det(C_{i,b}^{i,c})$ where $C_{i,b}^{i,c}$ is the matrix where we dropped the ith column and the row c from $C_{i,b}$. The matrix $C_{i,b}^{i,c}$ is a square submatrix of $\delta[\mathcal{K}]$, so we can bound

$|\det(C_{i,b})| \leq \mathcal{T}(\mathcal{K})$. Thus, $r_i = \det(C_{i,b})/\det(C) \leq \sqrt{n} \cdot \mathcal{T}_{\max}(\mathcal{K})$ and $\|r\| = \sqrt{\sum_{i=1}^{m} r_i^2} \leq \sqrt{n^2 \cdot \mathcal{T}_{\max}(\mathcal{K})^2} = n \cdot \mathcal{T}_{\max}(\mathcal{K})$ The potential energy of r is $\|\delta[\mathcal{K}]r\|^2$. We can bound this using Lemma 40 to obtain $\|\delta[\mathcal{K}]r\|^2 = O\left(n^3 \cdot d \cdot \mathcal{T}_{\max}(\mathcal{K})^2\right)$. ◀

C Embedded complexes

In this section, we consider the special case when $\mathcal{K}$ is a d-dimensional simplicial complex with a given embedding into $\mathbb{R}^{d+1}$. In this case, $\mathcal{K}$ is an **embedded complex**. Embedded complexes serve as a high dimensional generalization of planar graphs and naturally admit a dual graph. More specifically, we will generalize the special case of planar graphs for which the vertices s and t appear on the boundary of the same face. Throughout this section we assume we are given the embedding as input. Computing the dual graph from an embedding can be done in polynomial time [6]. We will show that the effective capacitance of a $(d-1)$-dimensional cycle γ in $\mathcal{K}$ is equal to the effective resistance between a pair of vertices that are "dual" to γ. Hence, we can parameterize the quantum algorithm deciding if γ is null-homologous in terms of the effective resistance in $\mathcal{K}$ and the effective resistance in the dual graph of $\mathcal{K}$. This section generalizes the analysis of planar graphs given by Jeffery and Kimmel [13]. The setup for our analysis has appeared in the author's previous work [20].

The Alexander duality theorem [10, Corollary 3.45], states that for a d-dimensional simplicial complex $\mathcal{K}$ with an embedding into $\mathbb{R}^{d+1}$ the subspace $\mathbb{R}^{d+1} \setminus \mathcal{K}$ consists of $\beta_d + 1$ connected components where β_d is the dimension of $H_d(\mathcal{K})$. We call these connected components **voids** and exactly one of these voids is unbounded. We denote the bounded voids as V_i for $1 \leq i \leq \beta_d$ and the unbounded void as V_∞. Moreover, the boundaries of the bounded voids generate the homology group $H_d(\mathcal{K})$. The embedding implies that each d-simplex is contained on the boundary of at most two voids, and we make the assumption that the d-simplices are oriented consistently with respect to the voids. That is, if a d-simplex is on the boundary of two voids it is oriented positively on one void, and negatively on the other. We have a boundary matrix ∂_{d+1} whose columns are the voids and whose rows are the d-simplices. From the embedding and the consistent orientation we see that ∂_{d+1} is the edge-vertex incident matrix of the **directed dual graph**: the directed graph whose vertices are in bijection with the voids and whose edges are in bijection with the d-simplices of $\mathcal{K}$. The direction of the edges are inherited from the orientations of the d-simplices. For a d-simplex σ on the boundary of voids V_1 and V_2 we denote the dual edge by $\sigma^* = (v_1^*, v_2^*)$ and we define the dual weight function by $w^*(\sigma^*) = 1/w(\sigma)$.

We construct an additional chain group $C_{d+1}(\mathcal{K})$ whose basis elements are the bounded voids. This is a purely algebraic construction and gives rise to a new chain complex $\cdots \rightarrow C_{d+1}(\mathcal{K}) \xrightarrow{\partial_{d+1}} C_d(\mathcal{K}) \xrightarrow{\partial_d} \ldots \xrightarrow{\partial_1} C_0(\mathcal{K})$. Since the boundaries of the voids generate the dth homology group of $\mathcal{K}$ and $C_{d+1}(\mathcal{K})$ is generated by these voids we obtain a valid chain complex. Moreover, we have that $\dim H_d(\mathcal{K}) = 0$ in our new chain complex. More generally, we define the **dual complex** of $\mathcal{K}$, denoted $\mathcal{K}^*$, by the isomorphism $C_{d-k+1}(\mathcal{K}) \cong C_k(\mathcal{K})$. That is, the $(d-k+1)$-simplices of $\mathcal{K}$ are in bijection with the k-simplices of $\mathcal{K}^*$. The dual graph is the 1-skeleton of $\mathcal{K}^*$. Moreover, we define the dual boundary operator $\partial_k^* \colon C_k(\mathcal{K}^*) \rightarrow C_{k-1}(\mathcal{K}^*)$ to be the coboundary operator $\delta_{d-k+1} \colon C_{d-k+1}(\mathcal{K}) \rightarrow C_{d-k}(\mathcal{K})$ of $\mathcal{K}$, and the dual coboundary operator $\delta_k^* \colon C_{k-1}(\mathcal{K}^*) \rightarrow C_k(\mathcal{K}^*)$ to be the boundary operator $\partial_{d-k+1} \colon C_{d-k+1}(\mathcal{K}) \rightarrow C_{d-k}(\mathcal{K})$ of $\mathcal{K}$. In other words the (co)boundary operators commute with the duality isomorphism. We summarize the construction in Figure 4 in the appendix.

We need to make one additional assumption on the location of the input $(d-1)$-dimensional cycle γ which makes our setup a generalization of a planar graph with two vertices s and t appearing on the same face. To achieve this we assume that there exists a void V_i with two unit

γ-flows Γ_1 and Γ_2 such that $\text{supp}(\Gamma_1) \cap \text{supp}(\Gamma_2) = \emptyset$ and $\text{supp}(\Gamma_1) \cup \text{supp}(\Gamma_2) = \text{supp}(\partial_{d+1} V_i)$. That is, there exist two unit γ-flows whose supports partition the boundary of the void V_i. This generalizes the fact in planar graphs that when s and t are on the same face we can find two st-paths which partition the boundary of the face. In planar graphs we are guaranteed to find two such paths, however for an arbitrary $(d-1)$-cycle γ we are not guaranteed to find two unit γ-flows partitioning the boundary of some void. More specifically, we take Γ_2 to be a unit $(-\gamma)$-flow so that $\partial_d \Gamma_2 = -\gamma$. In the planar graph analogy this is equivalent as viewing Γ_1 as a path from s to t and viewing Γ_2 as a path from t to s. We add an additional basis element Σ to $C_d(\mathcal{K})$ such that $\partial_d \Sigma = -\gamma$. In planar graphs this is equivalent to adding an edge directed from t to s. In a planar graphs the addition of this edge splits the face containing s and t into two. In higher dimensions the geometry is more complicated, but the addition of Σ allows us to perform a purely algebraic operation makes our chain complex behave as if V_i has been split into two. We remove V_i from $C_{d+1}(\mathcal{K})$ and replace it with two new basis elements V_s and V_t. Next, we extend the boundary operator to V_s and V_t in the following way: $\partial_{d+1} V_s = \Gamma_1 - \Sigma$ and $\partial_{d+1} V_t = \Gamma_2 + \Sigma$. In the dual complex the vertices dual to V_s and V_t are denoted s^* and t^*. Dual to Σ is an edge $\Sigma^* = (t^*, s^*)$. In the next section we will show that the effective capacitance of γ in a subcomplex $\mathcal{K}(x)$ is equal to the effective resistance between s^* and t^* in the subgraph of the dual graph which is the 1-skeleton of $\mathcal{K}^*(x)$. Note that the 1-skeleton of $\mathcal{K}^*(x)$ contains all of the vertices of $\mathcal{K}^*$ but only includes the edges dual to the d-simplices in $\mathcal{K}(x)$.

Effective capacitance is dual to effective resistance

The effective resistance between s^* and t^* in $\mathcal{K}^*(x)$ is determined by the unit s^*t^*-flows in $\mathcal{K}^*(x)$. However, it will be convenient to work with **circulations** instead of flows. A unit s^*t^*-circulation f is a cycle; that is, an element of $\ker \partial_1^*$, such that $f(\Sigma^*) = 1$. Recall that Σ^* is the edge directed from t^* to s^*, so a unit s^*t^*-circulation is just a unit s^*t^*-flow with the additional edge Σ^* completing the cycle. Clearly, there is a bijection between unit s^*t^*-flows and unit s^*t^*-circulations and we define the flow energy of a circulation to be equal to the flow energy of its corresponding flow.

▶ **Theorem 41.** *Let $\mathcal{K}$ be a d-dimensional simplicial complex embedded into $\mathbb{R}^{d+1}$, and let γ be a $(d-1)$-cycle such that there exist two unit γ-flows Γ_1 and Γ_2 whose supports partition the boundary of some void V_i. The effective capacitance $\mathcal{C}_\gamma(\mathcal{K}(x))$ is equal to the effective resistance $\mathcal{R}_{s^*t^*}(\mathcal{K}^*(x))$.*

Proof. Let p be a unit γ-potential in $\mathcal{K}(x)$ and we define f to be the image of δp under the duality isomorphism; that is, $f = \partial_2^* p^*$, which makes f a circulation in the 1-skeleton of $\mathcal{K}^*(x)$. Further, since $\Sigma^* = (t^*, s^*)$ the circulation f corresponds to a unit s^*t^*-flow by the following calculation: $f^T \Sigma^* = p^T(\partial_d \Sigma) = p^T \gamma = 1$. Next, we calculate the flow energy of f and show it is equal to the potential energy of p.

$$
\begin{aligned}
\mathsf{J}(f) &= \sum_{e^* \in \mathcal{K}^*(x)_1} \frac{f(e^*)^2}{w^*(e^*)} \\
&= \sum_{e^* \in \mathcal{K}^*(x)_1} f(\sigma^*)^2 w(e) \\
&= \sum_{\sigma \in \mathcal{K}(x)_d} ((\delta p)^T \sigma)^2 w(\sigma) \\
&= \mathcal{J}(p)
\end{aligned}
$$

Conversely, let f^* be a unit s^*t^*-circulation in $\mathcal{K}^*(x)$. By the assumptions outlined in the beginning of the section we have $\dim H_d(\mathcal{K}) = 0$ which in turn gives us $\dim H_1(\mathcal{K}^*) = 0$. Hence, f^* can be written as a linear combination of boundaries $f^* = \sum \alpha_i B_i$ where $B_i \in \operatorname{im} \partial_2^*$. Let p^* be the 2-chain in $\mathcal{K}^*(x)$ with $\partial_2^* p^* = f^*$; we will show that p is the unit γ-potential in $\mathcal{K}(x)$ in bijection with f^*. To see that p is a unit γ-potential we compute its inner product with γ:

$$
\begin{aligned}
p^T \gamma &= p^T (\partial_d \Sigma) \\
&= (\delta_{d-1} p)^T \Sigma \\
&= (\partial_2^* p^*)^T \Sigma^* \\
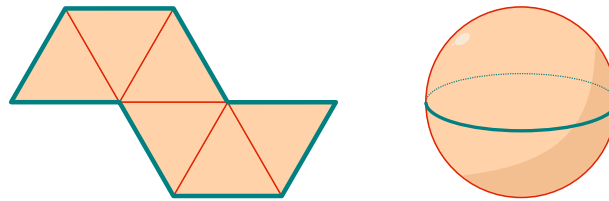&= (f^*)^T \Sigma^* \\
&= 1.
\end{aligned}
$$

It remains to show that the potential energy of p is equal to the flow energy of $|f^*\rangle$. We have the following calculation:

$$
\begin{aligned}
\mathcal{J}(p) &= \sum_{\sigma \in \mathcal{K}(x)_d} ((\delta_{d-1} p)^T \sigma)^2 w(\sigma) \\
&= \sum_{\sigma \in \mathcal{K}(x)_d} \frac{((\delta_{d-1} p)^T \sigma)^2}{w^*(\sigma^*)} \\
&= \sum_{\sigma \in \mathcal{K}(x)_d} \frac{((\partial_2^* p^*)^T \sigma^*)^2}{w^*(\sigma^*)} \\
&= \sum_{\sigma \in \mathcal{K}(x)_d} \frac{f^*(\sigma)^2}{w^*(\sigma^*)} \\
&= \mathsf{J}(f^*). \qquad\blacktriangleleft
\end{aligned}
$$

D   Figures
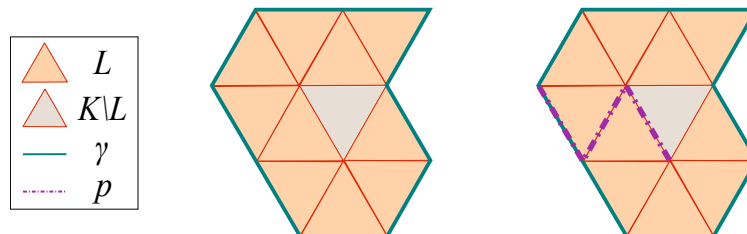
Figure 1 Left: A 1-cycle γ. Right: A unit γ-potential p. If this complex is unweighted, then the potential energy of p is 1.

Figure 2 Left: The unique unit γ-flow is the 6 triangles in series. If the complex is unweighted, then the effective resistance of γ is 6. Right: The cycle γ is the equator of the sphere, and the two hemispheres are two unit γ-flows in parallel. If each hemisphere has potential energy 1, then the effective resistance of γ is $\frac{1}{2}$.
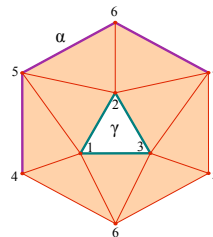
Figure 3 The complex $\mathbb{RP}_\gamma$. The inner cycle is γ, and half of the outer cycle is α. The boundary of the sum of the triangles is $2\alpha + \gamma$.

$$C_{d+1}(\mathcal{K}) \xleftarrow[\delta_d]{\partial_{d+1}} C_d(\mathcal{K}) \xleftarrow[\delta_{d-1}]{\partial_d} \ldots \xleftarrow[\delta_1]{\partial_1} C_0(\mathcal{K})$$

$$\Big\Updownarrow \cong \qquad \Big\Updownarrow \cong \qquad\qquad \Big\Updownarrow \cong$$

$$C_0(\mathcal{K}^*) \xleftarrow[\partial_1^*]{\delta_0^*} C_1(\mathcal{K}^*) \xleftarrow[\partial_2^*]{\delta_1^*} \ldots \xleftarrow[\partial_{d+1}^*]{\delta_d^*} C_{d+1}(\mathcal{K}^*)$$

Figure 4 The commutative diagram summarizing the dual complex construction.