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On the treewidth of Hanoi graphs

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ABSTRACT

The objective of the well-known *Tower of Hanoi* puzzle is to move a set of discs one at a time from one of a set of pegs to another, while keeping the discs sorted on each peg. We propose an adversarial variation in which the first player forbids a set of states in the puzzle, and the second player must then convert one randomly-selected state to another without passing through forbidden states. Analyzing this version raises the question of the *treewidth* of *Hanoi graphs*. We find this number exactly for three-peg puzzles and provide nearly-tight asymptotic bounds for larger numbers of pegs.

1. Introduction

The *Tower of Hanoi* puzzle is well known (for a comprehensive treatment see [1]), but it loses its fun once its player learns the strategy. It has some number n of discs of distinct sizes, each with a central hole allowing it to be stacked on any of three pegs. The discs start all stacked on a single peg, sorted from largest at the bottom to smallest at the top. They must be moved one at a time until they are all on another peg, while at all times keeping the discs in sorted order on each peg. The optimal strategy is easy to follow: alternate between moving the smallest disc to a peg that was not its previous location, and moving another disc (the only one that can be moved). Once one learns how to do this, and that the strategy takes $2^n - 1$ moves to execute [2], it becomes tedious rather than fun.

The puzzle can be modified in several ways to make it more of an intellectual challenge and less of an exercise in not losing one's place. One of the most commonly studied variations involves using some number p of pegs that may be larger than three. Of course, one can ignore the extra pegs, but using them allows shorter solutions. An optimal solution for four pegs was given by Bousch in 2014 [3], but the best solution for larger numbers of pegs remains open. The Frame–Stewart algorithm solves these cases, but it is not known if it is optimal [4]. The length of an optimal solution, for starting and ending positions of the discs chosen to make this solution as long as possible, can be modeled graph-theoretically using a graph called the *Hanoi graph*, which we denote H_p^n. This graph is formed by constructing a vertex for each configuration of the game, and connecting two vertices with an edge when their configurations are connected by one legal move. The number of moves between the two farthest-apart positions is then the diameter of this graph. For three pegs, the diameter

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of H_3^n is $2^n - 1$ (the traditional starting and ending positions are the farthest apart) but for $p > 3$ the diameter of H_p^n is unknown [5].

In this paper, we consider a different way of making the puzzle more difficult, by making it adversarial. In our version of the game, the first of two players selects a predetermined number of *forbidden positions*, that the second player cannot pass through. Then, the second player must solve a puzzle using the remaining positions. If that were all, then the first player could win by forbidding only a very small number of positions, the $p - 1$ positions one move away from the start position. To make the first player work harder, after the first player chooses the forbidden positions, we choose the start and end position randomly from among the positions in the game. We ask: How many positions must the first player forbid, in order to make this a fair game, one where both players have equal chances of being able to win?

We can model this problem graph-theoretically, as asking for the smallest number of vertices to remove from a Hanoi graph in order for the number of pairs of remaining vertices belonging within the same component as each other to be half the total number of pairs of vertices. The answer to the problem lies between the minimum size of a *balanced vertex separator* (Lemma 2) and (up to a constant factor of three) the minimum order of a *recursive balanced vertex separator*; the latter is equivalent, up to constant factors, to asking for the *treewidth* of H_p^n. (Technically, the treewidth can be larger than the recursive separator order by a logarithmic factor when this order is constant, but both are within constant factors of each other when the order is polynomial and not bounded by a constant.) Treewidth is of interest to computer scientists as many NP-hard graph problems become fixed-parameter tractable on graphs with bounded treewidth [6].

1.1. New results and prior work

We conjecture that the treewidth of H_p^n is $\Theta((p-2)^n)$. For $p > 3$ this bound is exponential, and we make progress towards this conjecture by proving that the treewidth is within a polynomial factor of this bound. More precisely we show an asymptotic upper bound of $O((p-2)^n)$ and an asymptotic lower bound of $\Omega(n^{-(p-1)/2} \cdot (p-2)^n)$. We increase the lower bound to $\Omega(\frac{2^n}{n})$ when $p = 4$. Moreover, we find the exact (constant) treewidth of H_3^n and of the closely-related Sierpiński triangle graphs. Our results provide an answer to our motivating question on sizes of forbidden sets of positions, up to polynomial factors for four or more pegs and exactly for three pegs.

As a byproduct of our proof techniques, we observe a nearly linear asymptotic lower bound on the treewidth of the *Kneser graph* (Corollary 2), whenever $n \geq 2k + 1$. Harvey and Wood [7] showed a previous exact result for the treewidth of $\mathrm{Kn}(n, k)$, requiring that n be at least quadratic in k. Another byproduct of our proof techniques gives an alternative proof (Lemma 18) that the treewidth of the *tensor product* $G \times H$ of two graphs G and H is at least t—where t is the treewidth of G—whenever H is not bipartite. (As Hickingbotham and Wood [8] have noted, Bottreau and Métivier [9] showed this fact using graph minor properties.) Eppstein and Havvaei [10] showed that the treewidth of $G \times H$ (where H need not be non-bipartite) is at most $s(t + 1) - 1$, where s is the number of vertices in H. Thus in the case of the tensor product of any graph G with a non-bipartite, bounded-size graph H, Lemma 18 brings the upper and lower bounds for the treewidth of $G \times H$ to within a constant factor of one another.

Brevšar and Spacapan [11] gave analogous results respectively for edge connectivity; Weichsel [12] and Sinha and Garg [13] gave similar results for connectedness; Kozawa, Otachi, and Yamazaki [14] gave lower bounds for the treewidth of the strong product and Cartesian product of graphs.

2. Preliminaries

2.1. Hanoi graphs

We consider the Tower of Hanoi puzzle with n discs and p pegs, numbered from 1 to n and from 1 to p respectively. (Traditionally, $p = 3$.) Label the n discs, in order of increasing size, as $d_1, \ldots, d_n$. If discs d_i and d_j are on the same peg, and $i < j$, then d_j is constrained to be below d_i. A legal move in the game consists of moving the top (smallest) disc on some peg A to another peg B, while preserving the constraint. At the beginning of the game, all n discs are on the first peg. The objective of the game is to obtain, through some sequence of legal moves, a state in which all n discs are on the last peg.

Formally, a *configuration* of the p-peg, n-disc Tower of Hanoi game is an n-tuple $(p_1, p_2, \ldots, p_n)$ where $p_i \in \{1, 2, \ldots, p\}$, describing the peg for each disc d_i. We say two configurations $(p_1, p_2, \ldots, p_n)$ and $(p'_1, p'_2, \ldots, p'_n)$ are *compatible* if a move from one configuration to the other is allowed. This happens exactly when the two configurations differ only in the value of a single coefficient p_i, for which d_i is the smallest disc having either of the two differing values. We call a configuration with all discs on a single peg a *perfect state*. The *Hanoi graph* H_p^n is a graph whose vertices are the configurations of the n-disc, p-peg Tower of Hanoi game, with an edge for each compatible pair of configurations. It has p^n vertices and $\frac{1}{2}\binom{p}{2}(p^n - (p-2)^n)$ edges [15].

2.2. Sierpiński triangle graphs

The *Sierpiński triangle graphs* $\{S_p^n\}$ are a family of graphs that have a planar embedding whose limit $n \to \infty$ is the *Sierpiński triangle* fractal when $p = 3$. For the case $p = 3$, the Hanoi graphs $\{H_3^n\}$ are known to be closely related to the

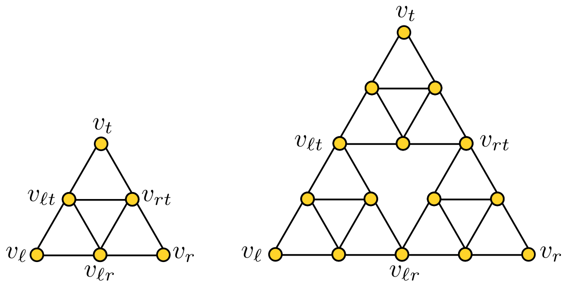

Fig. 1. The Sierpiński triangle graphs S_3^2 and S_3^3.

Sierpiński triangle graphs $\{S_3^n\}$. See Hinz, Klavžar, and Zemljič [16] for an extensive survey of these graphs and their properties, including their *Hamming dimension*. Klavžar, Peterin, and Zemljič [17] also analyzed the Hamming dimension of S_p^n. Barrière, Comellas, and Dalfó [18] analyzed the properties of a *small-world* variation on Sierpiński triangle graphs.

We present an inductive definition of the Sierpiński triangle graphs, along with a planar embedding of them. See Fig. 1 for the embeddings of S_3^2 and S_3^3. (Technically the graph we define is slightly different from that considered in [16] and [17]; their graph has edges that our S_3^n does not, and is in fact isomorphic to H_3^n.) The planar embedding will allow us to see the geometric similarity between the Sierpiński triangle graphs and the three-peg Hanoi graphs. The first Sierpiński triangle, S_3^1, is isomorphic to K_3 with a planar embedding of an equilateral triangle with unit length sides. The vertices of the triangle coincide with the vertices of K_3.

Inductively, we assume S_3^{n-1} has a planar embedding whose outer face is embedded geometrically as an equilateral triangle. We label the vertices on the outer face of the triangle v_ℓ, v_r, v_t which are the left, right, and top vertices, respectively. To construct S_3^n from S_3^{n-1} we take three copies of S_3^{n-1} labeled $S_3^{n-1,L}, S_3^{n-1,R}, S_3^{n-1,T}$ for the left, right, and top triangles and make the following vertex identifications.

1. Identify v_ℓ in $S_3^{n-1,R}$ with v_r in $S_3^{n-1,L}$, and call the resulting vertex $v_{\ell r}$.
2. Identify v_t in $S_3^{n-1,L}$ with v_ℓ in $S_3^{n-1,T}$, and call the resulting vertex $v_{\ell t}$.
3. Identify v_t in $S_3^{n-1,R}$ with v_r in $S_3^{n-1,T}$, and call the resulting vertex v_{rt}.

The resulting graph has a planar embedding whose outer face can again be embedded as a subdivided equilateral triangle. In S_3^n the left, right, and top vertices of the outer face are contained in $S_3^{n-1,L}$, $S_3^{n-1,R}$, and $S_3^{n-1,T}$ respectively. As before we denote them as v_ℓ, v_r, and v_t.

2.3. Recursive balanced separators, treewidth, and havens

In this section we give a brief discussion of the concepts of recursive balanced separators, treewidth, and havens. Given a graph $G=(V,E)$ a *vertex separator* is a subset $X \subseteq V$ such that $G \setminus V$ consists of two disjoint sets of vertices A and B with $A \cup B = V \setminus X$ and for all $a \in A$, $b \in B$ there is no edge (a,b) in the graph $G \setminus X$. Further, given a constant c with $\frac{1}{2} \le c < 1$, we call X a *balanced vertex separator* if $(1-c)|V| \le |A| \le \frac{|V|}{2}$ and $\frac{|V|}{2} \le |B| \le c|V|$. When this holds we call X a *c-separator*. We say that G has a *recursive balanced separator* of order s whenever either $|V| \le 1$, or we can find a balanced separator of size s for G, and the resulting subgraphs A and B have recursive balanced separators of order at most s respectively.

A *tree decomposition* of a graph G is a tree T whose nodes are sets of vertices in G called *bags*, such that the following conditions hold.

- If two vertices are adjacent, then they share at least one bag.
- If a vertex v is in two bags A and B, then v is in every bag on the path from A to B in T.
- Every vertex in $V(G)$ is in some bag.

The *width* of a tree decomposition T is one less than the maximum size of a bag in T. The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$, is the minimum width over all tree decompositions of G. The bags in the tree decomposition T induce vertex separators in G. Moreover, we can use the tree decomposition to find a recursive balanced separator for G. Hence, the treewidth of G is a measure of the minimum order of a recursive balanced separator for G. The following folklore lemma relates the order of a recursive balanced separator to the treewidth of a graph; see [19] and [20, Lemma 6.6].

Lemma 1. *With respect to every $\frac{1}{2} \le c < 1$, for every $d > 0$, there exists $k > 0$ such that the following hold for every graph G with N vertices, for sufficiently large N:*

1. *If $t = \mathrm{tw}(G)$, then with respect to c, G has a recursive balanced separator of order at most $k(t+1)$.*

2. *If G has a recursive balanced separator of order at most t, where* $t \geq N^d$*, then G has treewidth at most* kt.

Returning to our motivating game, in which one player forbids the use of a designated set of states in the state space of a puzzle and the other player attempts to connect two randomly chosen states by a path, we see that a fair number of states to forbid is controlled by the size of a recursive balanced separator. We formalize this in the following lemma:

Lemma 2. *Given a graph G, let* $f(G)$ *be the minimum number of vertices that can be removed from G so that, if two random vertices of G are chosen, the probability that they are not removed and have a path between them is at most* $1/2$*. Let* $c = 1/\sqrt{2}$*, and let* $r(G)$ *be the minimum size of a c-separator (not necessarily recursive) for G. Let s be the minimum order of a recursive c-separator for G. Then* $r(G) \leq f(G) \leq 3s$.

Proof. If we remove a vertex set X with $|X| = f(G)$, leaving probability less than $1/2$ that two randomly-chosen vertices from G are connected, then the remaining subgraph cannot contain any connected component larger than $|V(G)|/\sqrt{2}$. If it contains any connected component of size at least $|V(G)|/2$, then f separates that subgraph from the remaining vertices, and otherwise the remaining small subgraphs can be combined to give a separation between two subgraphs whose largest size is at most $2|V(G)|/3$, better than c. Therefore, $r(G) \leq f(G)$.

To show that $f(G) \leq 3s$, find a recursive c-separator for G of order s; the separator X has the following three separators as subsets: a c-separator X for G resulting in two separated subgraphs, and c-separators Y and Z for each of the two separated subgraphs. $|X| + |Y| + |Z| \leq 3s$. Removing $X \cup Y \cup Z$ from G partitions the rest of G into subgraphs of size at most $|V(G)|/2$. No matter which of these subgraphs one of the randomly chosen two vertices belongs to, the probability that the other vertex belongs to the same component will be at most $1/2$. □

Some of our results will bound the treewidth of graphs using *havens*, a mathematical formalization of an escape strategy for a robber in cop-and-robber pursuit-evasion games. In these games, a set of cops and a single robber are moving around on a given graph G. Initially the robber is placed at any vertex of the graphs, and none of the cops has been placed. In any move of the game, one of the cops can be removed from the graph, or a cop that has already been removed can be placed on any vertex of the graph. However, before the cop is placed, the robber (knowing where the cop will be placed) is allowed to move along any path in the graph that is free of other cops. The goal of the cops is to place a cop on the same vertex as the robber while simultaneously blocking all escape routes from that vertex, and the goal of the robber is to evade the cops forever. In these games, a *haven of order k* describes a strategy by which the robber can perpetually evade k cops, by specifying where the robber should move for each possible move by the cops. It is defined as a function ϕ, mapping each set of vertices $X \subseteq V$ with $|X| \leq k$ to a nonempty connected component in $G \setminus X$, such that whenever $X_1 \subseteq X_2$, $\phi(X_2) \subseteq \phi(X_1)$. A robber following this strategy will move to any vertex of $\phi(X)$, where X denotes the set of vertices to be occupied by the cops at the end of the move. The mathematical properties of havens ensure that the robber can always reach one of these vertices by a cop-free path.

Returning again to our adversarial version of the Tower of Hanoi puzzle, the cops-and-robber game is equivalent to a game in which the first player attempts to pin the second player to a state from which no legal move to any non-forbidden state is possible. The placement (or removal) of a cop is equivalent to the first player designating (or de-designating) a state as forbidden; an evasion strategy for a robber is equivalent to the existence of a legal move for the second player.

The existence of a haven in G yields a lower bound on the treewidth of G via the following lemma.

Lemma 3 *(Seymour and Thomas [21]). A graph G has a haven of order greater than or equal to k if and only if* $\mathrm{tw}(G) \geq k - 1$.

3. Three pegs

In this section we show that $\mathrm{tw}(H_3^n) \leq 4$ for all $n \geq 1$. We prove this by relating the three-peg Tower of Hanoi game and the Sierpiński triangle graph S_3^n. S_3^n has treewidth at least 3 for all n, as it contains a triangle, and (Lemma 4) it equals 4 for $n > 4$. Additionally, each Sierpiński triangle graph contains a smaller three-peg Hanoi graph as a minor, and vice versa. From this it will follow that $\mathrm{tw}(H_3^n) = 4$ for all sufficiently large n. For completeness we include a more detailed proof of the bounds on $\mathrm{tw}(S_3^n)$.

Note that we can recursively decompose S_3^n into a triangle and a trapezoid, from which the trapezoid further decomposes into two additional triangles. (Here, we only consider trapezoids whose long side is horizontal.) This recursive decomposition leads to the construction of a tree decomposition of S_3^n. The six distinguished vertices $v_\ell, v_r, v_t, v_{\ell r}, v_{\ell t}$, and v_{rt} define the bags of the tree decomposition at each level. The set $\{v_t, v_{\ell t}, v_{rt}, v_\ell, v_r\}$ lies on the perimeter of a triangle in this decomposition. We call a bag in the tree decomposition consisting of these vertices a *triangular bag*. On the other hand, the set $\{v_{\ell t}, v_{rt}, v_\ell, v_{\ell r}, v_r\}$ lies on the perimeter of a trapezoid in the decomposition. We call a bag in the tree decomposition consisting of these vertices a *trapezoidal bag*. With this definition we are now ready give a proof of the fact that $\mathrm{tw}(S_3^n) = 4$ for all $n > 4$.

Lemma 4. *The treewidth of* S_3^n *is equal to 4 for all* $n > 4$.

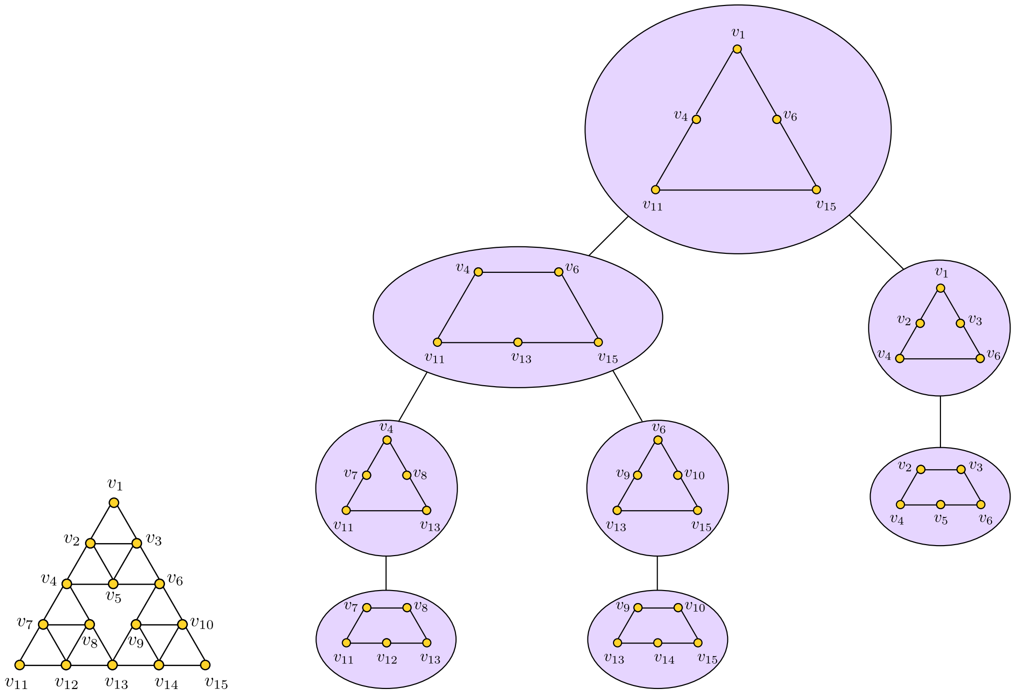

Fig. 2. S_3^3 along with its tree decomposition.

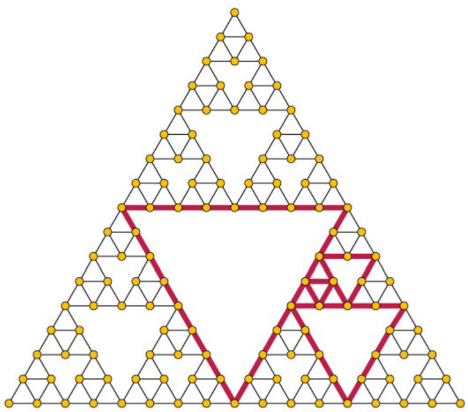

Fig. 3. The graph S_3^5 with a subdivision of the octahedral graph highlighted in red. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Proof. To prove the upper bound we construct a tree decomposition of S_3^n out of the triangular and trapezoidal bags defined above. We take the triangular bag in S_3^n to be the root of the tree decomposition, and recursively decompose S_3^n into its triangular and trapezoidal subgraphs. A bag at depth k is either a triangular or trapezoidal bag from an S_3^{n-k} subgraph. The children of a trapezoidal bag at depth k are the triangular bags corresponding to the two copies of S_3^{k-1} that make up the trapezoid. The children of a triangular bag at depth k are a trapezoidal and a triangular bag corresponding to the decomposition of S_3^k into a trapezoid and triangle. Every edge of S_3^n is contained in some triangle or trapezoid, and every triangle and trapezoid appear as a bag in the tree decomposition. For any vertex v in S_3^n if $v \in B_1, B_2$ where B_1 and B_2 are distinct bags there are two cases to consider. If B_1 is an ancestor of B_2 then v, by the construction of the bags, must be in every triangular or trapezoidal bag lying in between them. If there is no ancestry relationship, then v must lie in the intersection of the shapes defined by B_1 and B_2. Hence, there is some triangle or trapezoid containing both B_1 and B_2 which is their least common ancestor in the tree decomposition. See Fig. 2 for an illustration on S_3^3.

To prove the lower bound it is sufficient to show that S_3^n contains a subdivision of the octahedron graph when $n > 4$. The octahedron graph is a forbidden minor for treewidth 3 graphs [22]. See Fig. 3 for an illustration. □

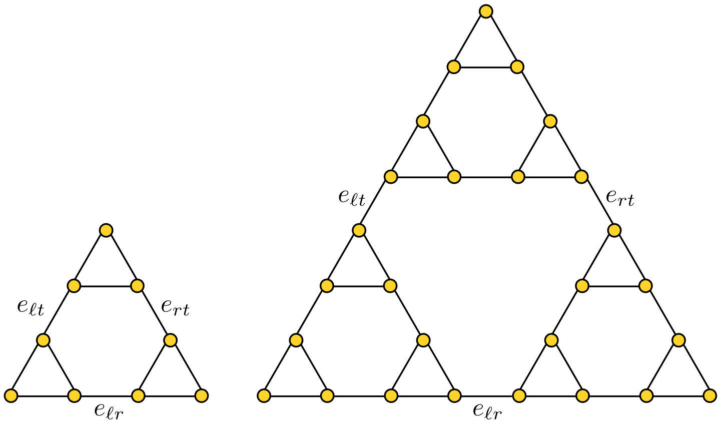

Fig. 4. The Hanoi graphs H_3^2 and H_3^3. We label the boundary edges such that their index coincides with their corresponding vertex in the Sierpiński triangle.

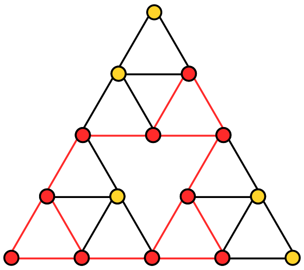

Fig. 5. S_3^3 with an H_3^2 minor highlighted in red.

Next we give an inductive construction of the Hanoi graph H_3^n with 3 pegs and n discs. This construction is almost identical to that of S_3^n, but instead of identifying vertices we connect the three copies of H_3^{n-1} with three edges. Recall that the vertices of H_3^n are configurations representing the game state, that is a vertex is an element of $\{1, 2, 3\}^n$. We define H_3^1 to be K_3 with the same planar embedding as in the case of the Sierpiński triangle and denote the vertices as the 1-tuples $(1), (2), (3)$. The cyclic ordering of the vertices does not affect our construction.

By induction we assume H_3^{n-1} has a planar embedding whose outer face is an equilateral triangle such that the corners of the triangle are the configurations corresponding with the perfect states, and we denote these vertices p_1, p_2, p_3. For $i \in \{1, 2, 3\}$ let H_i be the graph isomorphic to H_3^{n-1} with the vertex set $V(H_3^{n-1}) \times \{i\}$. We construct H_3^n out of the three H_i's and add the following edges.

1. Add an edge between $(p_1, 2)$ and $(p_1, 3)$ and denote it $e_{\ell r}$.
2. Add an edge between $(p_2, 1)$ and $(p_2, 3)$ and denote it e_{rt}.
3. Add an edge between $(p_3, 1)$ and $(p_3, 2)$ and denote it $e_{\ell t}$.

We call these three edges the *boundary edges*. The boundary edges represent the legal moves obtained by moving the largest disc. It is clear from the construction that the resulting graph embeds into the plane as an equilateral triangle with the perfect states at the corners of the triangle. See Fig. 4.

Theorem 1. $\mathrm{tw}(H_3^n) = 4$ *for all* $n > 4$.

Proof. To prove the lower bound we contract the boundary edges of H_3^n to create an S_3^n-minor. Hence, $4 = \mathrm{tw}(S_3^n) \le \mathrm{tw}(H_3^n)$ for $n > 4$.

To get the inequality $\mathrm{tw}(H_3^n) \le 4$ we inductively construct an H_3^n-minor of S_3^{n+1} as follows. For the base case we can easily find a copy of K_3 in S_3^2. Let G_1, G_2, G_3 be the three S_3^n subgraphs used to construct S_3^{n+1} and let $v_{i,j}$ be the vertex shared by G_i and G_j. By the inductive hypothesis we assume each G_i contains an H_3^{n-1}-minor which we denote by H_i. We construct an H_3^n-minor in S_3^{n+1} by connecting the corresponding perfect states of H_i and H_j via a path containing $v_{i,j}$ for each $i \ne j$. These paths can be chosen to be vertex-disjoint, which proves the theorem. See Fig. 5 for an illustration. □

The three-peg case is simple enough that we can analyze our forbidden-state version of the puzzle directly. If two states are forbidden, the only way to separate the remaining states is to separate one recursive subgraph of the same type from the rest of the graph. In terms of the original puzzle, the two forbidden states can be described by choosing a peg and a number k and forbidding the two states where the largest k discs are on the chosen peg and the remaining $n-k$ discs are all on the same peg as each other (one of the other two pegs). The probability of a connection between two randomly-chosen states is maximized for $k = 1$, for which, for large n, the probability of a path between two randomly-chosen vertices becomes approximately $(2/3)^2 + (1/3)^2 = 5/9$. On the other hand, if three states are forbidden, it becomes possible to separate the state space into three equally-sized subgraphs by forbidding three of the six states in which the largest disc can move. For this selection, the probability of a path between two randomly-chosen vertices becomes $3(1/3)^2 = 1/3$.

4. More pegs

We conjecture that the treewidth of the Hanoi graph H_p^n is $\Theta((p-2)^n)$. By Lemma 1 the same bound would automatically apply to the recursive balanced separator orders of these graphs; by Lemma 2, this would imply an upper bound on the number of states to forbid to make the adversarial version of the Hanoi puzzle fair ($f(G)$ in Lemma 2). In this section we make progress towards this conjecture by proving the asymptotic upper bound $\operatorname{tw}(H_p^n) = O((p-2)^n)$ and the asymptotic lower bound $\operatorname{tw}(H_p^n) = \Omega(n^{-(p-1)/2} \cdot (p-2)^n)$. We obtain the lower bound by proving that every balanced separator of H_p^n (recursive or otherwise) is of this asymptotic order. This lower bound then applies to $f(G)$ in Lemma 2. Our bounds are almost tight, off only by the factor $\Theta(n^{(p-1)/2})$. We begin by proving the asymptotic upper bound, which we do by constructing a recursive balanced separator of the required order and applying Lemma 1.

Theorem 2. *For any fixed $p \geq 3$ and $n \geq 1$, $\operatorname{tw}(H_p^n) = O((p-2)^n)$.*

Proof. We can recursively decompose H_p^n into p vertex-disjoint copies of H_p^{n-1} by considering the subgraphs induced by fixing the position of the largest disc in the configurations. We call a vertex a boundary vertex if in its configuration there is at least one peg occupied by no discs and the largest disc shares its peg with no other discs. These are the configurations in which the largest disc is free to move. The endpoints of edges between the H_p^{n-1} subgraphs in our decomposition are the boundary vertices.

We compute the order of our recursive balanced separator by counting the number of boundary vertices. This is the number of ways to distribute $n-1$ discs across $p-2$ pegs, hence the size of the separator is $\binom{p}{2}(p-2)^{n-1}$. Our choice of separator splits H_p^n into p subgraphs of size $\frac{1}{p}|V(H_p^n)|$. By grouping the H_p^{n-1} subgraphs into two vertex sets, we obtain a c-separator where $c \in \{\frac{\lceil p/2 \rceil}{p}, \frac{\lceil p/2 \rceil + 1}{p}, \ldots, \frac{p-1}{p}\}$ depending on our choice of vertex sets. Each subgraph can then be recursively decomposed in a similar way, and the number of vertices required in each recursive decomposition at level i is equal to $\binom{p}{2}(p-2)^{n-i}$. The theorem follows directly from Lemma 1. □

To prove the asymptotic lower bound we construct a new graph related to H_p^n whose treewidth is easier to compute. We can specify the positions of a subset of discs in a Hanoi puzzle by a mapping $\rho \colon [n] \to [p] \cup \{\infty\}$, where a finite value of $\rho(i)$ specifies the peg containing disc d_i and an infinite value means that disc d_i is allowed to be placed on any peg that does not also contain a specified disc. We define the *pegset* induced by ρ to be the states consistent with this specification. More formally, a vertex $v = (p_1, p_2, \ldots, p_n)$ is in the pegset induced by ρ if and only if:

1. for all $k \in [n]$, if $\rho(k) \neq \infty$ then $\rho(k) = p_k$, and
2. for all $k, l \in [n]$, if $\rho(k) = \infty \neq \rho(l)$, then $p_k \neq p_l$.

If $\rho(k) \neq \infty$ we call d_k *frozen* by ρ; further, if a peg p_k is in the image of ρ we call p_k frozen by ρ as well. Intuitively, a pegset is the result of freezing a set of discs onto a set of pegs and playing a Hanoi puzzle using only the remaining unfrozen discs and pegs. We are interested in pegsets that meet two additional properties:

3. exactly $p-3$ elements of $[p]$ have a non-empty inverse under ρ, and
4. for $j \in [p]$ either $|\rho^{-1}(j)| = \lfloor \frac{n-1}{p-2} \rfloor$ or $|\rho^{-1}(j)| = 0$.

We call such pegsets *regular pegsets*. Note that, because we still have three pegs unfrozen, and because the three-peg Hanoi graphs are connected, each regular pegset describes a connected subgraph of the Hanoi graph.

To make our analysis cleaner we assume that $n \equiv 1 \mod (p-2)$, hence properties 3 and 4 imply that there are precisely $\frac{n-1}{p-2} + 1$ unfrozen discs in a regular pegset. Note that this restriction on n does not change the overall asymptotic analysis for other values of n, as we can still lower-bound the treewidth for other n by rounding n down to a value with this restricted form.

Let I_p^n denote the graph whose vertices are the regular pegsets of H_p^n where two vertices share an edge if and only if the intersection of their corresponding pegsets is non-empty. We call I_p^n the *pegset intersection graph* of H_p^n. We characterize the adjacency condition in terms of frozen discs and pegs in Lemma 5.

Lemma 5. *Two regular pegsets u and v are adjacent in I_p^n if and only if the following criteria are satisfied:*

1. *if a disc is frozen by both u and v, then both u and v freeze it to the same peg,*
2. *u and v each freeze exactly one peg unfrozen by the other,*
3. *if a disc is frozen by u but not by v, then the peg it is frozen on is not frozen by v, and*
4. *if a disc is frozen by v but not by u, then the peg it is frozen on is not frozen by u.*

Proof. If u and v are adjacent in I_p^n then there exists some vertex contained in both pegsets, say $w = (w_1, w_2, \ldots, w_n)$. By the definition of a regular pegset both u and v freeze $\frac{n-1}{p-2}$ discs evenly across $p-3$ pegs and leave $\frac{n-1}{p-2}+1$ discs unfrozen. We will now show that each of the four claims follows from the adjacency of u and v.

1. Suppose for a contradiction that a disc d_i is frozen to different pegs by u and v; then no configuration in u can equal a configuration in v since they differ at the ith component.

2. If u and v freeze the same set of pegs then a configuration in u cannot equal a configuration in v since they will differ on the components corresponding to frozen discs. Now, assume u freezes more than one peg left unfrozen by v. A vertex w in the intersection of u and v would have a configuration that matches both u and v on their frozen discs, but the total number of discs frozen by u and v is at least $(p-1) \cdot \frac{n-1}{p-2}$; then w has more than n discs, contradicting the fact that w is a valid configuration.

3. Let u freeze the disc d_i onto the peg p_k and assume v does not freeze d_i. If v also freezes p_k then v must freeze $\frac{n-1}{p-2}$ discs onto p_k while leaving d_i unfrozen, hence there is no configuration in v that places d_i onto p_k.

4. Identical to 3.

We are now ready to prove the converse. Let u and v be pegsets in I_p^n such that conditions 1 through 4 hold. Conditions 1 and 2 tell us that the configurations of u and v coincide with one another for $(p-4) \cdot \frac{n-1}{p-2}$ discs evenly distributed across $p-4$ pegs. The remaining $2 \cdot \frac{n-1}{p-2}$ discs are left unfrozen by either u or v; call the set of these discs U. Conditions 3 and 4 ensure that we can choose a peg that is frozen by either u or v, but not both, and place $\frac{n-1}{p-2}$ of the discs in U onto this peg which yields a configuration shared by both u and v. □

As a consequence of Lemma 5 we can describe how to traverse an edge from a pegset u to a pegset v in I_p^n by freezing and unfreezing discs. We place $\frac{n-1}{p-2}$ of the discs left unfrozen by u onto the peg frozen by v but left unfrozen by u. Then, we take the peg frozen by u and left unfrozen by v and unfreeze every disc on it.

We will derive the asymptotic lower bound on $\mathrm{tw}(H_p^n)$ by first obtaining an asymptotic lower bound on $\mathrm{tw}(I_p^n)$. To compute the treewidth of I_p^n, we will take advantage of a prior result: Babai and Szegedy [23] showed (Lemma 6 below) that every *vertex-transitive* graph with sufficiently small diameter has large *vertex expansion*; it is easy to show that large vertex expansion in turn implies large treewidth. We define the relevant notions, and then show (i) that I_p^n is vertex transitive, and (ii) that I_p^n has small diameter.

Definition 1. A *graph automorphism*, defined over the vertex set V of a graph $G = (V, E)$, is a function $\phi : V \to V$ such that for all $u, v \in V$, $(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in E$.

That is, a graph automorphism is a permutation of the vertices of G that preserves the structure of G.

Definition 2. A graph $G = (V, E)$ is *vertex transitive* if for every $u, v \in V$, there exists a graph automorphism $\phi : V \to V$ such that $\phi(u) = v$.

Intuitively, vertex transitivity means that all vertices in G carry the same "information:" there is no special information that distinguishes any one vertex from the rest of the verices, other than its label.

Definition 3. The vertex expansion of a graph G is equal to

$$\min_{S \subseteq V(G): 1 \le |S| \le \frac{|V(G)|}{2}} \frac{|\partial S|}{|S|},$$

where ∂S is the union of the neighborhoods, in $G \setminus S$, of vertices in S.

Roughly speaking, small vertex expansion means that there exist small (not necessarily balanced) vertex separators in G, and large vertex expansion means that G does not have such separators. Since a lack of small separators implies a lack of small balanced separators, a lower bound on expansion yields a lower bound on treewidth. We will state this fact more precisely in the proof of Lemma 9.

We now state the result of Babai and Szegedy that we will use:

Lemma 6 *(Babai and Szegedy [23]). Let G be a vertex-transitive graph. Then the vertex expansion of G is $\Omega(1/d)$, where d is the diameter of G.*

We now show that I_p^n satisfies the conditions that allow us to apply Lemma 6:

Lemma 7. *I_p^n is vertex transitive.*

Proof. We define a family of automorphisms $\phi_{i,j}$ which swap the roles of d_i and d_j in some pegset. The lemma follows from the fact that we can transform a pegset u to any other pegset v by a sequence of swap operations. For any pegset u we define the image of u under $\phi_{i,j}$ to be

$$\phi_{i,j}(u)(k) = \begin{cases} u(i) & k = j \\ u(j) & k = i \\ u(k) & \text{otherwise} \end{cases}.$$

Let u and v be adjacent pegsets in I_p^n. By U_u and U_v we denote the sets of discs left unfrozen by u and v, respectively. By Lemma 5 there are pegs p_u and p_v such that u freezes discs onto p_u but not p_v, and v freezes disc onto p_v but not p_u. Further, traversing the edge from u to v is equivalent to placing $\frac{n-1}{p-2}$ discs from U_u onto p_k and treating the discs frozen to p_u as unfrozen. If u and v are adjacent then $\phi_{i,j}(u)$ and $\phi_{i,j}(v)$ are also adjacent, since swapping the labels of two discs does not affect the traversal process. If $\phi_{i,j}(u)$ and $\phi_{i,j}(v)$ are adjacent then so are u and v, by the above and the fact that $\phi_{i,j} = \phi_{i,j}^{-1}$. □

Lemma 8. *The diameter of I_p^n is $\Theta(n)$.*

Proof. Let u and v be pegsets in I_p^n. Let $k = \frac{n-1}{p-2}$. If u and v do not freeze the same set of pegs, we can, by freezing and unfreezing discs, walk along a path in I_p^n of length depending only on p, to a configuration that does freeze the same set of pegs as v, and continue with the process below. Therefore, assume u and v do freeze the same set of pegs, and label this set of pegs in increasing order by index as $Q = \{q_1, q_2, \ldots, q_{p-3}\}$. For $i = 1, \ldots, p-3$, let U_i be the set of discs frozen on q_i by u, and let V_i be the set frozen on q_i by v.

For all $i = 1, \ldots, p-3$, we iteratively transform u into v, one peg at a time. For a given peg q_i, the process for transforming U_i into V_i is as follows. There are three cases:

1. There exists some disc $d \in V_i \setminus U_i$ that is unfrozen by u,
2. There exists some disc $d \in V_i \setminus U_i$ that is frozen on some other peg by u,
3. or $U_i = V_i$.

In case 1, unfreeze q_i, then freeze an arbitrary peg $q_l \notin Q$, to obtain a new pegset w adjacent to the current pegset. Since each pegset leaves $\frac{n-1}{p-2} + 1$ discs unfrozen, let the new pegset freeze onto q_l all but one of the discs unfrozen by u. Let the omitted disc, d, be one in V_i that is unfrozen by u. Choose some $d' \in U_i \setminus V_i$ (one exists since $|U_i| = |V_i|$), then unfreeze q_l; freeze onto q_i the set $(U_i \setminus \{d'\}) \cup \{d\}$, to obtain a new adjacent pegset where d' is replaced by d.

Repeat this process until case 1 no longer applies, i.e. until every remaining $d \in V_i \setminus U_i$ is frozen by u. Then (case 2) consider some such d. u does not freeze d on a peg q_r to which this process has already been applied, since all such pegs now agree with v. Therefore, u freezes d on some peg q_s to which the process has not yet been applied. Unfreeze q_s and freeze an arbitrary unfrozen peg q_l to obtain the next pegset in the process; when doing so, some unfrozen disc d'' remains unfrozen. Then again freeze q_s, but omit d and instead freeze d'' onto q_s. d is now unfrozen, and we proceed as in case 1. Repeat case 2 until case 3 applies.

Repeating the overall process for every peg gives a path from u to v of length $O(n)$. □

Lemmas 7 and 8 now allow us to apply Lemma 6 to obtain a lower bound of $\Omega(\frac{1}{n}V(|I_p^n|))$ on the vertex expansion—and therefore the treewidth—of I_p^n:

Lemma 9. *The treewidth of I_p^n is $\Omega(\frac{1}{n}|V(I_p^n)|)$.*

Proof. By applying Lemmas 7, 8, and 6, along with the definition of vertex expansion, we have $|\partial S| = \Omega(\frac{|S|}{n})$ for all $S \subseteq V(I_p^n)$ with $0 \le |S| \le \frac{|V(I_p^n)|}{2}$, which implies that the order of any balanced vertex separator of I_p^n is bounded from below by $\Omega(\frac{|V(I_p^n)|}{n})$. It follows by Lemma 1 that the treewidth of I_p^n is also bounded from below by $\Omega(\frac{|V(I_p^n)|}{n})$. □

We now count the number of pegsets in $V(I_p^n)$.

Lemma 10. *The number of regular pegsets in I_p^n is $\Theta(n^{-(p-3)/2} \cdot (p-2)^n)$.*

Proof. There are $\binom{p}{p-3}$ ways to choose the frozen pegs. Each pegset divides the discs into $p-2$ sets of (almost) equal size and there are $\frac{n!}{((\frac{n}{p-2})!)^{p-2}}$ ways to choose the sets. This is because there are $n!$ ways to order the discs, but we only care about the ordering of the $p-2$ partitions of the discs, hence we divide by $\left(\frac{n}{p-2}!\right)^{p-2}$. (Asymptotically, we may assume $n \equiv 0 \mod p-2$.) In total there are $\binom{p}{p-3} \cdot \frac{n!}{((\frac{n}{p-2})!)^{p-2}}$ pegsets. Since p is fixed, we apply Stirling's approximation to $\frac{n!}{((\frac{n}{p-2})!)^{p-2}}$ to obtain the result. □

By applying Lemmas 9 and 10 we obtain the following corollary.

Corollary 1. $\mathrm{tw}(I_p^n) = \Omega(n^{-(p-1)/2} \cdot (p-2)^n)$.

Next we show how to obtain a lower bound of $\mathrm{tw}(H_p^n)$ from Corollary 1. Since we have a lower bound on the treewidth of I_p^n, Lemma 3 guarantees the existence of a haven of a useful order. The idea behind Lemma 11 is to take a haven of order $\Omega(n^{-(p-1)/2} \cdot (p-2)^n)$ in I_p^n and modify it to create a haven of the same asymptotic order in H_p^n.

Lemma 11. $\mathrm{tw}(H_p^n) = \Omega(\mathrm{tw}(I_p^n))$.

Proof. Let $k = \mathrm{tw}(I_p^n) + 1 = \Omega(n^{-(p-1)/2} \cdot (p-2)^n)$. By Lemma 3, I_p^n has a haven of order k. Call this haven ϕ. Recall that a haven describes an evasion strategy for a robber in a cops-and-robbers game. Intuitively, if a robber can evade the cops in I_p^n, the same robber can also evade the cops in H_p^n by playing only on states that belong to pegsets of I_p^n and by paying attention only to which of those pegsets are occupied by at least one cop. We formalize this strategy below by constructing a haven for H_p^n from ϕ. Because a cop moving in H_p^n may simultaneously occupy a constant number of pegsets in I_p^n, the order of the haven we construct is a constant factor smaller than that of ϕ.

Every vertex in I_p^n corresponds to a pegset; every pegset corresponds to a set of configurations in the Tower of Hanoi game. Each of these configurations corresponds to a vertex in H_p^n. Define the function $f : \mathcal{P}(V(H_p^n)) \to \mathcal{P}(V(I_p^n))$, where for $X \subseteq V(H_p^n)$, $f(X)$ is the set of vertices in I_p^n whose corresponding pegsets contain configurations in X. Define $g : \mathcal{P}(V(I_p^n)) \to \mathcal{P}(V(H_p^n))$, such that for $X' \subseteq V(I_p^n)$, $g(X')$ is the set of all configurations belonging to pegsets in X'. Let $g(X') = \emptyset$ if $X' = \emptyset$.

Define $\psi : \mathcal{P}(V(H_p^n)) \to \mathcal{P}(V(H_p^n))$, such that $\psi(X)$ is the connected component containing $g(\phi(f(X)))$. To show that ψ is a haven of the desired order, it suffices to show that:

1. for all $X \subseteq V(H_p^n)$, $\psi(X)$ is well-defined—i.e. $g(\phi(f(X)))$ is connected and nonempty whenever $\phi(f(X))$ is nonempty,
2. for $Z \subseteq V(H_p^n)$, $\psi(Z) \subseteq \psi(X)$ whenever $X \subseteq Z$, and
3. $|X| = \Omega(f(X))$.

For (1), to see that $g(\phi(f(X)))$ is connected in $H_p^n \setminus X$, consider any pair of configurations $u, v \in g(\phi(f(X)))$. u and v belong to pegsets a and b (respectively) in $\phi(f(X))$. a has a path P to b in $\phi(f(X))$, since $\phi(f(X))$ is connected. Every vertex (pegset) w in this path corresponds to the set $W' = g(w) \subseteq g(\phi(f(X)))$ of all configurations belonging to the pegset w. $W' \cap X = \emptyset$, or else by the definition of f, w would be in $f(X)$, contradicting the fact that $w \in \phi(f(X))$. Furthermore, W' is connected, since it is isomorphic to H_3^d (where $d < n$). Also, every edge (w_1, w_2) in P corresponds to a vertex $w' \in g(\phi(f(X)))$ belonging to $W_1' = g(w_1)$ and $W_2' = g(w_2)$. Therefore, u has a path to v in $H_p^n \setminus X$, obtained by traversing an H_3^d copy W' for every vertex $w \in P$, and moving between H_3^d copies W' and W'' that intersect at a vertex in $g(\phi(f(X)))$ for every edge in P.

For (2), if $X \subseteq Z \subseteq V(H_p^n)$, then $f(X) \subseteq f(Z)$. Since ϕ is a haven, $\phi(f(Z)) \subseteq \phi(f(X))$. Therefore, $g(\phi(f(Z))) \subseteq g(\phi(f(X)))$, and both $g(\phi(f(Z)))$ and $g(\phi(f(X)))$ are connected. If $\phi(f(Z)) = \emptyset$, then $\psi(X) = \emptyset$, and (2) is true. Therefore, suppose $\phi(f(Z)) \neq \emptyset$. Suppose for a contradiction that $\psi(Z) \nsubseteq \psi(X)$. Let u be a vertex in $\psi(Z) \cap \psi(X)$ (this intersection is nontrivial since it includes $g(\phi(f(Z)))$), and let v be a vertex in $\psi(Z) \setminus \psi(X)$. Suppose $(u, v) \in E(H_p^n)$. (Such a pair must exist because $\psi(Z)$ is connected.) Since $X \subseteq Z$,

$$u, v \in \psi(Z) \subseteq V(H_p^n) \setminus Z \subseteq V(H_p^n) \setminus X.$$

However, since $v \notin \psi(X)$, this contradicts that $\psi(X)$ is a connected component in $H_p^n \setminus X$.

(3) follows from the fact that $|f(X)| \leq (p-2)|X|$, since every vertex belongs to at most $p-2$ regular pegsets. □

We have now proven the second theorem of the section.

Theorem 3. *For any fixed* $p \geq 4$, $\text{tw}(H_p^n) = \Omega(n^{-(p-1)/2} \cdot (p-2)^n)$.

5. Four pegs

In this section we prove a lower bound on the treewidth of four-peg Hanoi graphs that, while still separated by a polynomial factor from the upper bound, is tighter than the one in Section 4. Theorems 2 and 3, together, give upper and lower bounds that differ by a polynomial factor in the number of discs of the Tower of Hanoi puzzle. Compared to the overall exponential size of the bound, this is a small gap, and it is tempting to try to close it further. The proof of Theorem 3 identifies the pegset intersection graph (I_p^n) as a hard part of the graph to separate, and leverages the vertex-transitive structure of this graph.

However, there are many configurations in the game that are ignored by focusing on the I_p^n graph: namely, all configurations where the numbers of discs on the pegs are arbitrary, i.e., not constrained to be equal to $\lfloor \frac{n}{p-2} \rfloor$ for $p-3$ of the pegs. We broaden our analysis of pegsets to prove the main result of this section:

Theorem 4. $\text{tw}(H_4^n) = \Omega(\frac{2^n}{n})$.

We begin by generalizing the pegset intersection graph beyond regular pegsets.

Definition 4. Let G_4^n be a graph whose vertices are the pegsets of H_4^n that freeze only one peg and that freeze at most $\lfloor \frac{n-1}{2} \rfloor$ discs onto that peg. In this graph, let vertices u and v be adjacent whenever the pegsets u and v freeze mutually disjoint sets of discs, and freeze them onto separate pegs.

Clearly I_4^n is an induced subgraph of G_4^n. We prove our improved bound by analyzing the relationship between G_4^n and the *Kneser graph*.

Definition 5 *(Lovasz [24]).* Let $[n] = \{1, \ldots, n\}$ be an indexing of the objects in an arbitrary set. The Kneser graph, denoted $\text{Kn}(n, k)$, is the graph whose vertices correspond to the k-element subsets of $[n]$, and whose edges are the pairs of vertices whose corresponding subsets are disjoint.

We restrict our attention to Kneser graphs that are connected, namely the graphs $\text{Kn}(n, k)$ where $n \geq 2k+1$.

The condition on disjoint subsets in the definition of Kneser graphs is analogous to the condition on disjoint subsets of pegs in the definition of G_4^n. (In fact, for any given $k \leq \lfloor \frac{n-1}{2} \rfloor$, the pegsets that freeze exactly k discs induce as a subgraph of G_4^n the tensor product of $\text{Kn}(n, k)$ with a 4-clique—see Definition 7.) However, G_4^n also includes a separate condition, of having different frozen pegs. An additional complication is that G_4^n allows sets of different sizes rather than only considering sets of a single size k. To account for all set sizes appropriately, we introduce an extension of the Kneser graph:

Definition 6. Let the disjoint subset graph, denoted $\text{Ds}(n, r)$, be the graph whose vertices are identified with the subsets $s \subseteq [n]$ with $|s| \leq r$, and whose edges are the pairs of vertices whose corresponding subsets are disjoint.

For convenience, we let $\text{Ds}(n) = \text{Ds}(n, \frac{n-1}{2})$. Clearly $|V(\text{Ds}(n))| \approx 2^{n-1}$. Then $V(G_4^n)$ consists of four copies of $V(\text{Ds}(n))$, with pegsets u and v connected iff they are in different copies and they share an edge in $\text{Ds}(n)$. In Lemma 12 we bound the treewidth of $\text{Ds}(n)$, after which we will use the relationship between G_4^n and $\text{Ds}(n)$ to prove Theorem 4.

Lemma 12. $\text{tw}(\text{Ds}(n)) = \Omega(\frac{2^n}{n})$.

We defer the formal proof of Lemma 12 to later but outline a proof sketch below. The idea of the proof is to observe that $\text{Ds}(n)$ consists of $\frac{n-1}{2}$ Kneser graph "slices." We make observations analogous to those leading to Corollary 1: Kneser graphs are vertex transitive (Remark 2) and have diameter $O(n)$ (Lemma 13), implying that for all $0 \leq k \leq \frac{n-1}{2}$, $\text{tw}(\text{Kn}(n,k)) = \Omega(\frac{1}{n}|V(\text{Kn}(n,k))|)$ (Corollary 2). Since

$$|V(\text{Ds}(n))| = \sum_{k=0}^{\frac{n-1}{2}} |V(\text{Kn}(n,k))|,$$

Lemma 12 then follows if we can, intuitively, show that the Kneser slices are hard to separate from one another. We formalize this notion and show that it is true for most of the slices. The argument relies on the subset definitions of the Kneser graphs' vertices, and makes use of the *Kruskal–Katona Theorem* (Corollary 3).

We prove that given a balanced vertex separator X for $\mathrm{Ds}(n)$, either:

1. X contains a large number of the vertices in $\mathrm{Ds}(n)$ (at least an $\Omega(\frac{1}{n})$ factor), or
2. after removing X from $\mathrm{Ds}(n)$, there is still a large connected component in $\mathrm{Ds}(n)$, leading to a contradiction.

In the second case, we derive the contradiction as follows: we observe that after removing X from $\mathrm{Ds}(n)$, if case (1) does not hold, then most of the vertices of $\mathrm{Ds}(n)$ lie in Kneser slices that have large connected components, since their intersection with X contains too few vertices for a balanced separator. Call this set of Kneser slices $K_{conn}(X)$. We prove that every pair of subgraphs $G_k = \mathrm{Kn}(n,k)$ and $G_l = \mathrm{Kn}(n,l)$ in $K_{conn}(X)$ have large connected components A_k and A_l that share an edge. Therefore, these large connected components, together, form a large connected component in $\mathrm{Ds}(n) \setminus X$, from which we derive the desired contradiction.

Finally, we use our lower bound on the treewidth of $\mathrm{Ds}(n)$ to derive a lower bound on the treewidth of G_4^n, and in turn on the treewidth of H_4^n. We obtain the former by proving a more general claim about the treewidth of the tensor product of two graphs, and the latter by a proof analogous to that of Lemma 11.

We begin by showing the required lower bound on the treewidth of the Kneser graph. We use the following result of Valencia-Pabon and Vera:

Lemma 13 *(Valencia-Pabon and Vera [25]). If* $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$, *then the diameter of* $\mathrm{Kn}(n,k)$ *is* $\lceil \frac{k-1}{n-2k} \rceil + 1$.

Remark 1. When $k \le \frac{n-1}{2}$, the diameter in Lemma 13 is $O(n)$.

The following fact about Kneser graphs is well known; it also follows from a straightforward adaptation of the proof of Lemma 7.

Remark 2. All Kneser graphs are vertex transitive.

Combining Lemmas 6 and 13 with Remarks 1 and 2, and observing the relationship between vertex expansion, balanced separators, and treewidth (as we did in the proof of Lemma 9), gives the following corollary:

Corollary 2. *For all k with* $1 \le k \le \frac{n-1}{2}$, $\mathrm{tw}(\mathrm{Kn}(n,k)) = \Omega(\frac{1}{n}|V(\mathrm{Kn}(n,k))|)$, *and for every constant c, the minimum size of a c-separator in* $\mathrm{Kn}(n,k)$ *is* $\Omega(\frac{1}{n}|V(\mathrm{Kn}(n,k))|)$.

Before turning to the interfaces between the Kneser slices, we establish a threshold value such that most of the vertices of $\mathrm{Ds}(n)$ lie in $\mathrm{Kn}(n,k)$ slices with values of k exceeding this threshold. Restricting our attention (in Lemma 15) to these slices will allow us to prove the mutual connectedness of the large connected components in case (2).

Lemma 14. *For every constant β with $\frac{1}{2} < \beta < 1$, there exists a constant ε such that*

$$\lim_{n\to\infty} \frac{\sum_{k=\frac{n}{2}-\varepsilon\sqrt{n}}^{\frac{n}{2}} |V(\mathrm{Kn}(n,k))|}{|V(\mathrm{Ds}(n))|} \ge \beta.$$

Proof. Let $B(n,p)$ denote the binomial distribution parameterized with probability p. The standard deviation of $B(n,\frac{1}{2})$ is $\frac{\sqrt{n}}{2}$. If f is the probability mass function of $B(n,\frac{1}{2})$, then $f(k) = \frac{1}{2^n}\binom{n}{k}$.

Let X be a random variable distributed according to $B(n,p)$.

By Chebyshev's inequality,

$$Pr[|X - \frac{n}{2}| \ge \varepsilon\sqrt{n}] \le \frac{1}{4\varepsilon^2}.$$

Setting $\varepsilon = \frac{1}{2\sqrt{1-\beta}}$, so that $\beta = 1 - \frac{1}{4\varepsilon^2}$, yields the desired result, since

$$\sum_{k=\frac{n}{2}-\varepsilon\sqrt{n}}^{\frac{n}{2}} \binom{n}{k} = \frac{1}{2} \sum_{k=\frac{n}{2}-\varepsilon\sqrt{n}}^{\frac{n}{2}+\varepsilon\sqrt{n}} \binom{n}{k} = 2^{n-1} \cdot Pr[|X - \frac{n}{2}| \le \varepsilon\sqrt{n}] \ge 2^{n-1}(1 - \frac{1}{4\varepsilon^2}),$$

and since $|V(\mathrm{Ds}(n))| = 2^{n-1}$. (Technically, if n is even, then

$$|V(\mathrm{Ds}(n))| = 2^{n-1}(1 - o(1)).$$

In this case we may simply increase ε by an arbitrarily small amount, and the lemma holds.) $\square$

In Lemma 16 we will prove the existence of the large connected component from which the contradiction is derived in case (2) of the discussion following the statement of Lemma 12. We will use the following lemma:

Lemma 15. *Let $\varepsilon > 0$ be fixed. Suppose $\frac{n-1}{2} - \varepsilon\sqrt{n} \le l < k \le \frac{n-1}{2}$, and let A_k and A_l be subsets, respectively, of the vertices in the $\mathrm{Kn}(n,k)$ and $\mathrm{Kn}(n,l)$ subgraphs of $\mathrm{Ds}(n)$. Suppose further that $|A_k| > \frac{1}{2}|V(\mathrm{Kn}(n,k))|$ and $|A_l| > \frac{1}{2}|V(\mathrm{Kn}(n,l))|$. Then A_k and A_l share an edge.*

The proof of Lemma 15 is inspired by Harvey and Wood's lower-bounding proof for the treewidth of certain Kneser graphs [7]. The proof uses the *Kruskal–Katona Theorem* (Corollary 3), which provides a lower bound, given a collection $\mathcal{F}$ of k-element subsets of $[n]$, on the number of l-element subsets of $[n]$ that are subsets of sets in $\mathcal{F}$. The following formulation of the Kruskal-Katona theorem is due to Lovász (Frankl gave a short proof):

Theorem 5 *(Kruskal–Katona Theorem [26],[27]). Let $\mathcal{F}$ be a family of k-element subsets of $[n]$, and let $\mathcal{E}$ be the set of all $k-1$-element subsets of sets in $\mathcal{F}$. Then whenever $|\mathcal{F}| \ge \binom{m}{k}$, $|\mathcal{E}| \ge \binom{m}{k-1}$.*

Applying induction on $l = k-1, \ldots, 1$ to Theorem 5 implies the following corollary:

Corollary 3. *Let $\mathcal{F}$ be a family of k-element subsets of $[n]$, and let $\mathcal{E}$ be the set of all l-element subsets of sets in $\mathcal{F}$, where $1 \le l < k$. Then whenever $|\mathcal{F}| \ge \binom{m}{k}$, $|\mathcal{E}| \ge \binom{m}{l}$.*

Using Corollary 3, we prove Lemma 15:

Proof of Lemma 15. For every $v \in V(\mathrm{Ds}(n))$, view v as the k-size subset with which it is identified, and let $\overline{v}$ be the set complement of v.

Let $B_k = \{\overline{v} \mid v \in A_k\}$. Define a function δ_l mapping vertices in $\mathrm{Kn}(n,k)$ to their neighborhoods in $\mathrm{Kn}(n,l)$: for all $v \in V(\mathrm{Kn}(n,k))$, let $\delta_l(v) = \{w \in V(\mathrm{Kn}(n,l)) \mid (v,w) \in E(\mathrm{Ds}(n))\}$.

Extend the domain of δ_l to sets of vertices in $\mathrm{Kn}(n,k)$: for all $Z \subseteq V(\mathrm{Kn}(n,k))$, let $\delta_l(Z) = \bigcup_{v\in Z} \delta_l(v)$.

Clearly, a vertex $u \in \mathrm{Kn}(n,l)$ is in $\delta_l(A_k)$ iff there exists some $w \in B_k$ such that, viewing the vertices in their combinatorial sense, $u \subseteq w$.

I.e., $\delta_l(A_k)$ consists precisely of the vertices that are identified with subsets of vertices in B_k. Since

$$|B_k| = |A_k| > \frac{1}{2}|\mathrm{Kn}(n,k)| = \frac{1}{2}\binom{n}{k} \ge \binom{n-1}{n-k},$$

Corollary 3 implies that

$$|\delta_l(A_k)| \ge \binom{n-1}{l} \ge \frac{1}{2}\binom{n}{l} \ge \frac{1}{2}|V(\mathrm{Kn}(n,l))|.$$

In the above inequalities we use the (easily verified) fact that whenever $i \le \frac{n}{2}$,

$$\binom{n-1}{n-i} \le \frac{1}{2}\binom{n}{i} \le \binom{n-1}{i}.$$

Since by assumption $|A_l| > \frac{1}{2}|V(\mathrm{Kn}(n,l))|$, this implies that $\delta_l(A_k) \cap A_l \ne \emptyset$. That is, some vertex in A_k shares an edge with some vertex in A_l. $\square$

We are ready to formalize case (2) (Lemma 16) in the discussion following the statement of Lemma 12.

Lemma 16. *Let X be a vertex separator for $\mathrm{Ds}(n)$. Let $\frac{1}{2} < c < 1$ and $\varepsilon > 0$ be constants. Let*

$$K_{big} = \{\mathrm{Kn}(n,k) | \frac{n-1}{2} - \varepsilon\sqrt{n} \le k \le \frac{n-1}{2}\}$$

be the largest $\varepsilon\sqrt{n}$ Kneser subgraphs of $\mathrm{Ds}(n)$. Let

$$K_{conn}(X) = \{\mathrm{Kn}(n,k) \in K_{big} | \frac{|X \cap V(\mathrm{Kn}(n,k))|}{|V(\mathrm{Kn}(n,k))|} < \frac{a}{n}\},$$

where $a > 0$ is a constant.

Then for a suitable choice of a, if n is sufficiently large, for all $\mathrm{Kn}(n,k) \in K_{conn}(X)$, $\mathrm{Kn}(n,k) \setminus X$ *has a connected component* A_k *of size at least* $c(1 - O(\frac{1}{n}))|V(\mathrm{Kn}(n,k))|$, *and for all* $l \neq k$, *if* $\mathrm{Kn}(n,l) \in K_{conn}(X)$, *then* A_k *and* A_l *share an edge.*

Proof. By Corollary 2, for all $\mathrm{Kn}(n,k) \in K_{conn}(X)$, the minimum c-separator size for $\mathrm{Kn}(n,k)$ is at least $\frac{b}{n}|V(\mathrm{Kn}(n,k))|)$ for some constant $b > 0$, which by assumption is more than the vertices of X that lie in $\mathrm{Kn}(n,k)$ — at least when n is sufficiently large and when $a < b$. This implies that A_k is of the stated size. For the second part of the claim, consider any A_k, A_l pair. A_k and A_l are connected by Lemma 15, since $c(1 - O(\frac{1}{n})) \geq d$ for every constant d such that $c \geq d > \frac{1}{2}$. The lemma follows. □

We are now ready to prove Lemma 12. We choose numerical values instead of symbols for some of the constants that appear in the proof to make the argument more intuitive, although there are other values that work.

Proof of Lemma 12. Choose any constant $\frac{1}{2} < c < \frac{4}{7}$. Let X be a c-separator for $\mathrm{Ds}(n)$.

We will show that either X contains many vertices from large Kneser slices (those in $K_{sep}(X)$, which we define below), or most (more than a factor of c) of the vertices of $\mathrm{Ds}(n) \setminus X$ lie in a large connected component, so that X is not a c-separator.

Let K_{big} be the set of $\mathrm{Kn}(n,k)$ subgraphs with $\frac{n-1}{2} - \varepsilon\sqrt{n} \leq k \leq \frac{n-1}{2}$, where ε is chosen so that $\frac{|V(K_{big})|}{|V(\mathrm{Ds}(n))|} \geq \frac{8}{9}$. (We choose $\frac{8}{9}$ to make the argument work for $c < \frac{4}{7}$.) Let

$$K_{sep}(X) = \{\mathrm{Kn}(n,k) \in K_{big} | \frac{|X \cap V(\mathrm{Kn}(n,k))|}{|\mathrm{Kn}(n,k)|} \geq \frac{a}{n}\},$$

where a is chosen as in the proof of Lemma 16, according to the lower bound given by Corollary 2 on the minimum $\frac{5}{7}$-separator size for $\mathrm{Kn}(n,k) \in K_{big}$. (We choose $\frac{5}{7}$ because it produces the desired result for $c < \frac{4}{7}$.)

Let $K_{conn}(X) = K_{big} \setminus K_{sep}(X)$. There are two cases:

1. $\frac{|V(K_{sep}(X))|}{|V(K_{big})|} \geq \frac{1}{10}$.
2. $\frac{|V(K_{conn}(X))|}{|V(K_{big})|} > \frac{9}{10}$.

(We choose $\frac{1}{10}$ and $\frac{9}{10}$, again to make the argument work for $c < \frac{4}{7}$.)

In case 1, since $K_{sep}(X)$ is defined so that $\frac{|X \cap V(K_{sep}(X))|}{|V(K_{sep}(X))|} \geq \frac{a}{n}$,

$$\frac{|X \cap V(K_{sep}(X))|}{|V(\mathrm{Ds}(n))|} \geq \frac{|V(K_{sep}(X))|}{|V(K_{big})|} \cdot \frac{|V(K_{big})|}{|V(\mathrm{Ds}(n))|} \cdot \frac{a}{n} \geq \frac{1}{10} \cdot \frac{8}{9} \cdot \frac{a}{n} = \Omega(\frac{a}{n}) = \Omega(\frac{1}{n}).$$

In this case we are done.

In case 2, Lemma 16 implies that there exists a connected component A_k in every $\mathrm{Kn}(n,k) \subseteq K_{conn}(X)$ of size at least $(\frac{5}{7} - O(\frac{1}{n}))|V(\mathrm{Kn}(n,k))|$, and that every pair A_k and A_l are mutually connected. This implies that $\mathrm{Ds}(n) \setminus X$ has a connected component A such that

$$\frac{|V(A)|}{|V(\mathrm{Ds}(n))|} \geq (\frac{5}{7} - O(\frac{1}{n}))\frac{|V(K_{conn}(X))|}{|V(\mathrm{Ds}(n))|} \geq (\frac{5}{7} - O(\frac{1}{n}))(\frac{9}{10})(\frac{|V(K_{big})|}{|V(\mathrm{Ds}(n))|}$$
$$\geq (\frac{5}{7} - O(\frac{1}{n}))(\frac{9}{10})(\frac{8}{9}) = \frac{4}{7} - O(\frac{1}{n}) > c.$$

This contradicts the assumption that X is a c-separator for $\mathrm{Ds}(n)$. □

To show that $\mathrm{tw}(H_4^n) = \Omega(\mathrm{tw}(\mathrm{Ds}(n))$, we first show that the treewidth of the generalized pegset intersection graph G_4^n defined earlier is at least that of $\mathrm{Ds}(n)$, then that $\mathrm{tw}(H_4^n) = \Omega(\mathrm{tw}(G_4^n))$. Both of these are accomplished via haven mappings (Lemmas 17 and 19) of a similar flavor to Lemma 11.

Lemma 17. $\mathrm{tw}(G_4^n) = \Omega(\mathrm{tw}(\mathrm{Ds}(n)))$.

We prove Lemma 17 as a special case of a more general claim, Lemma 18, about the treewidth of the *tensor product* of graphs:

Definition 7. The tensor product of graphs G and H, denoted $G \times H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$, and whose edges are the pairs of (u_1, v_1) and (u_2, v_2) whose first and second components share edges in $E(G)$ and $E(H)$ respectively, i.e.

$$\{((u_1, v_1), (u_2, v_2)) \mid u_1, u_2 \in V(G), v_1, v_2 \in V(H), (u_1, u_2) \in E(G), (v_1, v_2) \in E(G)\}.$$

Bottreau and Métivier [9] proved that for every graph G and every non-bipartite graph H, G is a minor of $G \times H$. As Hickingbotham and Wood [8] have observed, this implies the following:

Lemma 18. *Let G and H be connected graphs, and suppose that H is not bipartite. Then*

$$\mathrm{tw}(G \times H) \geq \mathrm{tw}(G).$$

We give an alternative proof of Lemma 18 using havens. We first define an association between the vertices of G and those of $J = G \times H$.

Definition 8. Given the tensor product $J = G \times H$ of graphs G and H, define $f : V(J) \to V(G)$ so that for all $(u, v) \in V(J)$,

$$f((u, v)) = u.$$

Define $g : V(G) \to \mathcal{P}(V(J))$ so that for all $u \in V(G)$,

$$g(u) = f^{-1}(u) = \{(u, v) \mid v \in V(H)\}.$$

We use this definition to prove Lemma 18. The proof is similar in spirit to the proof of Lemma 11, and also uses the odd cycle property of bipartite graphs in a similar fashion to [9]:

Proof of Lemma 18. For the lower bound $\mathrm{tw}(G \times H) \geq \mathrm{tw}(G)$, by Lemma 3, G has a haven ϕ of order $k = \mathrm{tw}(G) + 1$. We construct a haven ψ in $J = G \times H$ of order $k' \geq k$, from which the lemma follows. To define ψ, we extend the domains of f and g to sets of vertices in the natural way. That is, for every $X \subseteq V(J)$, let $f(X)$ be the image of all vertices in X under f. For every $Y \in V(G)$, let $g(Y)$ be the union of the images under g of all vertices in Y.

For all $X \subseteq V(J)$, let $\psi(X)$ be the connected component in $J \setminus X$ containing $g(\phi(f(X)))$. (Let $\psi(\emptyset) = \emptyset$.) It suffices to show that:

1. $Y' = g(Y)$ is a nonempty connected component in $J \setminus X$ whenever Y is a nonempty connected component in $G \setminus f(X)$,
2. for all $X \subseteq Z \subseteq V(J)$, $\psi(Z) \subseteq \psi(X)$, and
3. for all $X \subseteq V(J)$, $|f(X)| \leq |X|$.

For (1), suppose Y is a connected component in $G \setminus f(X)$ for some $X \subseteq V(J)$. Let $Y' = g(Y)$. If $|Y| > 1$, then consider any edge $(u, w) \in Y$. Then for every pair of vertices $v, x \in V(H)$, the vertices (u, v) and (w, x) are connected by a path P' in Y'. To construct this path, consider any walk P along a sequence of vertices $(v, z_1, z_2, \ldots, z_l, x)$ of odd length in H from v to x. Such a walk must exist since H is not bipartite, i.e. contains an odd cycle. Construct the corresponding path P' in Y' by alternating between copies of u and copies of w. That is, let

$$P' = ((u, v), (w, z_1), (u, z_2), (w, z_3), \ldots, (w, z_{l-1}), (u, z_l), (w, x)).$$

Since such a path exists for every edge $(u, w) \in Y$, and Y is connected, Y' is also connected.

We deal with the degenerate case $|Y| = 1$ by letting Y' be a single copy (u, v) of the vertex $u \in Y$, and obtain ψ by extending this copy to a connected component.

For (2), it follows from the definition of f and the fact that ϕ is a haven, that $\phi(f(Z)) \subseteq \phi(f(X))$. ψ merely extends $\phi(f(Z))$ and $\phi(f(X))$ to connected components in $J \setminus Z$ and $J \setminus X$ respectively. The connected component B in $J \setminus X$ containing $\phi(f(Z))$ is the same as the connected component in $J \setminus X$ containing $\phi(f(X))$, since both $\phi(f(Z))$ and $\phi(f(X))$ are connected and one is a subset of the other. Furthermore, since $X \subseteq Z$, $J \setminus Z \subseteq J \setminus X$, so removing the additional vertices in $Z \setminus X$ from B cannot result in a connected component with vertices missing from B. That is, the connected component $\psi(Z)$ in $J \setminus Z$ containing $\phi(f(Z))$ is a subset of the connected component $\psi(X)$ in $J \setminus X$ containing $\phi(f(X))$.

(3) is immediate from the definition of f. □

Lemma 17 immediately follows from Lemma 18 and the fact that G_4^n is isomorphic to $\mathrm{Ds}(n) \times K_4$.

Lemma 19. $\mathrm{tw}(H_4^n) = \Omega(\mathrm{tw}(G_4^n))$.

Proof. Construct a haven mapping analogous to the mapping in Lemma 11. In Lemma 11 we defined f and g as, respectively, mapping sets of configurations to the regular pegsets to which they belong, and mapping sets of regular pegsets to the unions of their configurations. Extend the codomain of f and the domain of g, beyond regular pegsets, to the set of all pegsets in G_4^n. The rest of the argument is similar to the proof of Lemma 11. Again we need to check the following conditions:

1. for all $X \subseteq V(H_p^n)$, $\psi(X)$ is well-defined—i.e. $g(\phi(f(X)))$ is connected and nonempty whenever $\phi(f(X))$ is nonempty,
2. for $Z \subseteq V(H_p^n)$, $\psi(Z) \subseteq \psi(X)$ whenever $X \subseteq Z$, and
3. $|X| = \Omega(f(X))$.

(3) is easy since every configuration belongs to at most four pegsets. The reasoning for (1) is identical to that in the proof of Lemma 11. For (2), the reasoning is also the same. □

Theorem 4 follows from Lemma 12, Lemma 17, and Lemma 19.

6. Conclusion

Theorem 3 and Theorem 4, together with Theorem 1 and Theorem 2, give nearly tight asymptotic bounds on the number of states, in the adversarial version of the Tower of Hanoi game we proposed in the introduction, that the first player must forbid in order to ensure better than even odds of defeating the second player. This raises additional questions. First, suppose the first player forbids enough states to separate the graph in a balanced way, but the second player is fortunate enough to have starting and ending positions in the same connected component. What is the optimal strategy for the second player, and how many moves will this strategy take? Must this strategy be formulated in graph-theoretic terms, or is there an algorithm that consists of moving the discs in an intuitive way?

Theorem 4 (in Section 5) improves the lower bound of Theorem 3 when $p = 4$; one question would be to see whether the technique in the proof of Theorem 4 could be adapted to deal more generally with the structure of pegset intersection graphs when $p \geq 4$, yielding a bound of $\Omega(\frac{(p-2)^n}{n})$ in general when $p \geq 4$. However, this still would not eliminate the asymptotic gap between our upper and lower bounds.

To this end, Corollary 2 gives a lower bound on the treewidth of the Kneser graph that is new when $2k+1 \leq n \leq 3k-1$. Can this lower bound be tightened? Harvey and Wood [7] showed that in this case (when $2k+1 \leq n \leq 3k-1$),

$$\mathrm{tw}(\mathrm{Kn}(n,k)) < \binom{n-1}{k} - 1.$$

Since $\binom{n-1}{k} = \Theta(\binom{n}{k})$ when $2k+1 \leq n \leq 3k-1$, this upper bound does not imply sublinear treewidth. However, if in fact the treewidth is sublinear, and can be used to obtain a sublinear vertex separator in $\mathrm{Ds}(n)$ (defined in Section 5), then combined with a proof of asymptotic tightness in Lemma 17 and Lemma 19, this would imply that $\mathrm{tw}(H_4^n)$ is $o(2^n)$, proving that the upper bound in Theorem 2 is not tight. This would be surprising, as the family of separators given in Theorem 2 seems intuitively to target the “weakest” parts of the graph.

Another possible line of further research is whether the bound given in Lemma 18 is tight for the tensor product, and what can be said about other graph products. As stated in the introduction, Kozawa, Otachi, and Yamazaki [14] gave lower bounds for the Cartesian and strong products. Since the strong product of a graph has the same vertices as and a superset of the edges of the tensor product, our lower bound in Lemma 18 for the tensor product's treewidth immediately gives a lower bound on the treewidth of the strong product. However, Kozawa, Otachi, and Yamazaki [14] gave a stronger lower bound for the strong product. One question would be whether a comparable improvement over our bound can be proven for the tensor product.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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